Logical Consequence: Semantics

1 Truth Assignments

Definition 1 (Truth Assignment)
A **truth assignment** (i.e. an **interpretation**) is a function which takes a schema as input (including the atomic sentences "p", "q", "r", ...) and assigns that schema to exactly one truth value.

If v is a truth assignment, it must satisfy these rules (for all A and B):

$$\begin{aligned}
v(-A) = \top &\quad \text{iff} \quad v(A) = \bot \\
v(A \,.\, B) = \top &\quad \text{iff} \quad \text{both } v(A) = \top \text{ and } v(B) = \top \\
v(A \vee B) = \top &\quad \text{iff} \quad \text{either } v(A) = \top \text{ or } v(B) = \top \\
v(A \supset B) = \top &\quad \text{iff} \quad \text{either } v(A) = \bot \text{ or } v(B) = \top \\
v(A \equiv B) = \top &\quad \text{iff} \quad v(A) = v(B)
\end{aligned}$$

Informally: truth assignments are "ways the world could be."
Formally: truth assignments are just rows in a truth table.

The definition above can be formulated equally well in terms of which schemata v assigns $\bot$. For instance: $v(A \supset B) = \bot$ iff both $v(A) = \top$ and $v(B) = \bot$.

Exercise 2
Fill in the blanks:

$$\begin{aligned}
v(-A) = \bot &\quad \text{iff} \\
v(A \,.\, B) = \bot &\quad \text{iff} \\
v(A \vee B) = \bot &\quad \text{iff} \\
v(A \supset B) = \bot &\quad \text{iff} \\
v(A \equiv B) = \bot &\quad \text{iff}
\end{aligned}$$

OBSERVATION: What's the point of all this? Now you have *three* ways to determine when a complex TF-schema A is true:

(1) Write down a truth table for A (tedious).

(2) Use the equivalences from the previous handout or write the schema in disjunctive normal form.

(3) Determine what truth assignments make A true.

EXAMPLE 3

When is the schema "$-(p \vee (q \supset p))$" true? That is, under what interpretations of "p" and "q" is this schema true? Using the rules above, we reason as follows:

$$\begin{aligned} v\,(-(p \vee (q \supset p))) = \top \quad & \text{iff} \quad v(p \vee (q \supset p)) = \bot \\ & \text{iff} \quad \text{both } v(p) = \bot \text{ and } v(q \supset p) = \bot \\ & \text{iff} \quad \text{both } v(p) = \bot \text{ and } v(q) = \top \text{ and } v(p) = \bot \\ & \text{iff} \quad \text{both } v(p) = \bot \text{ and } v(q) = \top \end{aligned}$$

Hence, the schema "$-(p \vee (q \supset p))$" is true when both "p" is false and "q" is true; otherwise, the schema is false.

When is the schema false? Well:

$$\begin{aligned} v\,(-(p \vee (q \supset p))) = \bot \quad & \text{iff} \quad v(p \vee (q \supset p)) = \top \\ & \text{iff} \quad \text{either } v(p) = \top \text{ or } v(q \supset p) = \top \\ & \text{iff} \quad \text{either } v(p) = \top \text{ or } v(q) = \bot \text{ or } v(p) = \top \\ & \text{iff} \quad \text{either } v(p) = \top \text{ or } v(q) = \bot \end{aligned}$$

Hence the schema is false when either "p" is true or "q" is false; otherwise, it is true. Notice this is consistent with our previous answer.

Exercise 4
Try out "$-(p \,.\, (p \supset q)) \vee -q$":

$$v\,(-(p \,.\, (p \supset q)) \vee -q) = \top \quad \text{iff}$$

2 Satisfiability

Definition 5 (Satisfiability)
When a schema A is given the value $\top$ under *at least one* truth assignment, we say A is **satisfiable**. Otherwise, we say A is **unsatisfiable**.

Example 6
"$-(p \vee (q \supset p))$" is satisfiable: at least one interpretation makes it true, viz. one where $v(p) = \bot$ and $v(q) = \top$.

Example 7
"$-(p \supset q) \,.\, -(q \supset p)$" is unsatisfiable:

$$\begin{aligned} v\,(-(p \supset q) \,.\, -(q \supset p)) = \top \quad & \text{iff} \quad \text{both } v\,(-(p \supset q)) = \top \text{ and } v(-(q \supset p)) = \top \\ & \text{iff} \quad \text{both } v(p \supset q) = \bot \text{ and } v(q \supset p) = \bot \\ & \text{iff} \quad \text{both } v(p) = \top \text{ and } v(q) = \bot\text{, and moreover} \\ & \qquad \text{both } v(q) = \top \text{ and } v(p) = \bot \end{aligned}$$

But this can never happen: v can only assign "p" to *one* truth value (similarly for "q"). So "$-(p \supset q) \,.\, -(q \supset p)$" is unsatisfiable. (Weird, huh?)

3 Validity

Definition 8 (Validity)
When a schema A is true under *every* truth assignment, we say A is **valid**, or that A is a **tautology**.

Notation: If A is valid, we may write "$\models A$" to indicate this. If A is *not* valid (i.e. if A is false on some truth assignment), we may write "$\not\models A$" instead.

Warning: $\not\models A$ does *not* mean $\models -A$. For instance, let $A =$ "p".

Example 9
"$p \vee (p \supset q)$" is valid:

$$\begin{aligned} v(p \vee (p \supset q)) = \top \quad & \text{iff} \quad \text{either } v(p) = \top \text{ or } v(p \supset q) = \top \\ & \text{iff} \quad \text{either } v(p) = \top \text{ or } v(p) = \bot \text{ or } v(q) = \top \end{aligned}$$

But this will always happen: v must *alway* assign "p" to *some* truth value. So, "$p \vee (p \supset q)$" is valid. (Weird, huh?)

4 Implication

Definition 10 (Implication)
We say A **implies** B (or A **entails** B) if and only if every truth assignment v where $v(A) = \top$ is also a truth assignment where $v(B) = \top$. That is, A implies B if and only if there is no truth assignment v where $v(A) = \top$ but $v(B) = \bot$.

If A implies B, A is a **premise** and B is a **conclusion**.

Notation: We write "$A \models B$" to mean "A implies B." When A *doesn't* imply B, we write "$A \not\models B$."

WARNING: $A \not\models B$ does *not* mean $A \models -B$. For instance, let A = "p" and B = "q."

COMMENT: There are two ways to show that $A \models B$ using truth assignments:

(i) ***Going Forward***: Supposing $v(A) = \top$, show it must be that $v(B) = \top$.

(ii) ***Going Backward***: Supposing $v(B) = \bot$, show it must be that $v(A) = \bot$.

EXAMPLE 11 (GOING FORWARD)
$p \,.\, (q \vee r) \models (p \,.\, q) \vee (p \,.\, r)$:

Suppose $v(p \,.\, (q \vee r)) = \top$. We know that:

$v(p \,.\, (q \vee r)) = \top$ iff both $v(p) = \top$ and $v(q \vee r) = \top$
iff both $v(p) = \top$ and either $v(q) = \top$ or $v(r) = \top$

So $v(p) = \top$, and either $v(q) = \top$ or $v(r) = \top$. We want to show from this that $v((p \,.\, q) \vee (p \,.\, r)) = \top$. But:

$v((p \,.\, q) \vee (p \,.\, r)) = \top$ iff either $v(p \,.\, q) = \top$ or $v(p \,.\, r) = \top$
iff either both $v(p) = \top$ and $v(q) = \top$ or else both $v(p) = \top$ and $v(r) = \top$

So we must show that either $v(p) = v(q) = \top$, or $v(p) = v(r) = \top$, given what we already know about v. We already know that $v(p) = \top$, and we know that either $v(q) = \top$ or $v(r) = \top$.

- **Suppose that** $v(q) = \top$. Since we know that $v(p) = \top$, we can infer from this that both $v(p) = \top$ and $v(q) = \top$. Hence, $v((p \,.\, q) \vee (p \,.\, r)) = \top$.
- **Suppose instead** $v(r) = \top$. Again, since we know that $v(p) = \top$, we can infer that both $v(p) = \top$ and $v(r) = \top$. Hence, $v((p \,.\, q) \vee (p \,.\, r)) = \top$.

So either way, we have $v((p \,.\, q) \vee (p \,.\, r)) = \top$. □

EXAMPLE 12 (GOING BACKWARD)
$p \,.\, (q \vee r) \models (p \,.\, q) \vee (p \,.\, r)$: (same problem)

Suppose $v((p \,.\, q) \vee (p \,.\, r)) = \bot$. We know that:

$$\begin{aligned} v((p \,.\, q) \vee (p \,.\, r)) = \bot \quad & \text{iff} \quad \text{both } v(p \,.\, q) = \bot \text{ and } v(p \,.\, r) = \bot \\ & \text{iff} \quad \text{either } v(p) = \bot \text{ or } v(q) = \bot \text{, and moreover} \\ & \qquad\;\; \text{either } v(p) = \bot \text{ or } v(r) = \bot \end{aligned}$$

So either $v(p) = \bot$ or $v(q) = \bot$. Furthermore, either $v(p) = \bot$ or $v(r) = \bot$. We want to show from this that $v(p \,.\, (q \vee r)) = \bot$. But:

$$\begin{aligned} v(p \,.\, (q \vee r)) = \bot \quad & \text{iff} \quad \text{either } v(p) = \bot \text{ or } v(q \vee r) = \bot \\ & \text{iff} \quad \text{either } v(p) = \bot \text{ or} \\ & \qquad\;\; \text{both } v(q) = \bot \text{ and } v(r) = \bot \end{aligned}$$

So we must show that either $v(p) = \bot$, or else both $v(q) = \bot$ and $v(r) = \bot$. Apart from the above, we know (trivially) that either $v(p) = \top$ or $v(p) = \bot$.

- **Suppose** $v(p) = \bot$. Then we can infer that either $v(p) = \bot$ or both $v(q) = \bot$ and $v(r) = \bot$. So $v(p \,.\, (q \vee r)) = \bot$.
- **Suppose** $v(p) = \top$. Since we know that either $v(p) = \bot$ or $v(q) = \bot$, we can infer $v(q) = \bot$. Similarly, since we know that either $v(p) = \bot$ or $v(q) = \bot$, we can infer $v(r) = \bot$. So both $v(q) = \bot$ and $v(r) = \bot$. But from this, we can infer that either $v(p) = \bot$ or both $v(q) = \bot$ and $v(r) = \bot$. So $v(p \,.\, (q \vee r)) = \bot$.

So either way, we have $v(p \,.\, (q \vee r)) = \bot$. □

COMMENT: Showing that $A \not\models B$ is less systematic. To show $A \not\models B$, one must find a *counter-example*, i.e. an interpretation where A is true but B is false.

EXAMPLE 13 (COUNTER-EXAMPLE)
$p \supset q \not\models (p \vee r) \supset q$. Consider the following truth assignment:

$$v(p) = \bot$$
$$v(q) = \bot$$
$$v(r) = \top$$

According to this truth assignment, $v(p \supset q) = \top$. But $v(p \vee r) = \top$, since $v(r) = \top$, and yet $v(q) = \bot$. So $v((p \vee r) \supset q) = \bot$. Hence, $p \supset q \not\models (p \vee r) \supset q$.

5 Generalized Implication

DEFINITION 14 (GENERALIZED IMPLICATION)
Let $A_1, \ldots, A_n, B$ all be schemata. We say that $A_1, \ldots, A_n$ **imply** B if and only if every truth assignment v where $v(A_1) = v(A_2) = \cdots = v(A_n) = \top$, is also a truth assignment where $v(B) = \top$. If $A_1, \ldots, A_n$ imply B, we write $A_1, \ldots, A_n \models B$.

LEMMA 15
$A_1, \ldots, A_n \models B$ if and only if $(A_1 \,.\, (A_2 \,.\, (A_3 \,.\, \cdots \,.\, A_n) \cdots) \models B$ (i.e. if and only if the iterated conjunction of $A_1, \ldots, A_n$ implies B).

6 Equivalence

DEFINITION 16 (EQUIVALENCE)
We say that A and B are **equivalent** if and only if A and B are given the same truth value for any truth assignment. That is, A and B are equivalent if and only if for every truth assignment v, $v(A) = v(B)$.

NOTATION: We sometimes write "$A \dashv\models B$" to mean "A is equivalent to B." But we also sometimes write "$A \Leftrightarrow B$."

LEMMA 17
$A \dashv\models B$ if and only if both $A \models B$ and $B \models A$.

REMARK: Although the notation '$\dashv\models$' is suggestive, we cannot just *assume* that this Lemma is true. We must prove it. Thankfully, the proof isn't that complex.

▶ **PROOF:**

("only if" part)

Suppose $A \dashv\models B$. Then for every truth assignment v, $v(A) = v(B)$. So if $v(A) = \top$, then $v(B) = \top$; hence $A \models B$. Similarly, if $v(B) = \top$, then $v(A) = \top$; hence $B \models A$. □

("if" part)

Suppose $A \models B$ and $B \models A$. If $v(A) \neq v(B)$, then either $v(A) = \top$ and $v(B) = \bot$, or vice versa. But if $v(A) = \top$ and $v(B) = \bot$, then $A \not\models B$, contrary to supposition. Similarly, if $v(B) = \top$ and $v(A) = \bot$, then $B \not\models A$, again contrary to supposition. So it can't be that $v(A) \neq v(B)$. Thus $A \dashv\models B$. □

7 Implication vs. Conditional

Logical implication "$\models$" isn't the same as the conditional "$\supset$." For one, "$A \supset B$" is a schema, whereas "$A \models B$" is a relation between schemata. For another, if "$A \models B$" is true, then so is "$A \supset B$"; but even if "$A \supset B$" is true, it doesn't follow that "$A \models B$" is true. Consider:

"If Mal is the captain of Serenity, then he is a good captain."

This may be true, but it's not as though "Mal is the captain of Serenity" *logically implies* "Mal is a good captain." It is logically possible, after all, that Mal is a bad captain, even if he is captain of Serenity. Similarly,

"If Joey studies, then he'll pass."

may be *just so happen* to be true (because Joey happens to be a smart fellow), but it's certainly possible to imagine a scenario where Joey studies but doesn't pass. Thus, **one should never read "$A \supset B$" as "A *implies* B" or "A *entails* B." Similar remarks hold about reading the *material biconditional* "$A \equiv B$" as "A is equivalent to B."**

Despite all that, '$\models$' and '$\supset$' *do* have a close connection to one another.

THEOREM 18 (DEDUCTION THEOREM)
Let A and B be schemata. Then $A \models B$ if and only if $\models (A \supset B)$.

THEOREM 19 (GENERALIZED DEDUCTION THEOREM)
More generally, if $A_1, \ldots, A_n, B$ are schemata, then $A_1, \ldots, A_n \vDash B$ if and only if $A_1, \ldots, A_{n-1} \vDash (A_n \supset B)$.

COMMENT: What the Deduction Theorem shows is that there's still a strong connection between logical implication and the material conditional. The theorem says that "A implies B" if and only if "$A \supset B$" is *valid*, i.e. if and only if "$A \supset B$" *is a logical truth*. So while the material conditional doesn't *express* implication, there's a sense in which the material conditional still *indicates* it. This is what Goldfarb is referring to with the use/mention distinction for logical entailment.