Polyadic Logic: Semantics

DEFINITION 1 (INTERPRETATION)
An **interpretation** of polyadic schemata is a function that takes a schema as input and assigns that schema to exactly one truth value. It consists of:

(i)   a UD, i.e. some set of objects U (our "universe")

(ii) an extension assignment ext, which assigns each n-place predicate letter to a set of n-tuples over the UD

(iii) a variable assignment var, which assigns each (free) variable to a member of the UD.

(iv) a truth assignment val for the sentence letters

Recall from TF-logic: when giving an interpretation of a schema, you didn't need to specify the truth values of sentence letters not in the schema. For instance, if you're specifying an interpretation of "$p \supset q$," you don't need to specify the truth value of "r". You could, but it's not necessary (or important).

Similarly, when giving an interpretation of a polyadic schema, you don't necessarily need to specify each part of the schema. The rule is:

(i)   you must always specify the UD

(ii) you must specify the extension of every predicate letter occuring in your schema

(iii) you must specify the assignment of every variable that occurs *free* in your schema

(iv) you must specify the truth value of each sentence letter in your schema

NOTATION: If our UD is the set U, then U^2 is the set of ordered pairs $\langle a, b \rangle$ over U. That is, $U^2 = \{\langle a, b \rangle \mid a \in U \text{ and } b \in U\}$. Similarly, U^n is the set of ordered n-tuples $\langle a_1, \ldots, a_n \rangle$ over U. That is, $U^n = \{\langle a_1, \ldots, a_n \rangle \mid a_1, \ldots, a_n \in U\}$.

EXAMPLE 2
Let v be the following interpretation:

- $U = \{\text{Arc, Caitlin, Russ}\}$

- $\text{ext}(F) = \{\text{Caitlin}\}$, $\text{ext}(G) = \{\text{Caitlin, Russ}\}$

- $\text{ext}(R) = \{\langle\text{Russ, Arc}\rangle, \langle\text{Caitlin, Arc}\rangle, \langle\text{Arc, Arc}\rangle\}$

- $\text{var}(x) = \text{Russ}$, $\text{var}(y) = \text{Arc}$

- $\text{val}(p) = \bot$

Then the following schemata get the values below:

(i) $v(Fx) = \bot$

(ii) $v(Gy \supset Fy) = \top$

(iii) $v(Rxy) = \top$

(iv) $v(Ryx) = \bot$

(v) $v(\forall x \; Rxy) = \top$

(vi) $v(\forall y \; Rxy) = \bot$

(vii) $v(\forall x \; (\exists y \; (Gy . Ryx) \supset Fx)) = \bot$

(viii) $v(\forall x \; (-Rxx . Rxy \supset Gx)) = \top$

Recall that when interpreting 1-place predicates, we could sometimes draw circles to get an idea of what the world looks like. For instance, labeling a for "Arc", c for "Caitlin", and r for "Russ" (for convenience), the monadic part of the interpretation above can be drawn as so:
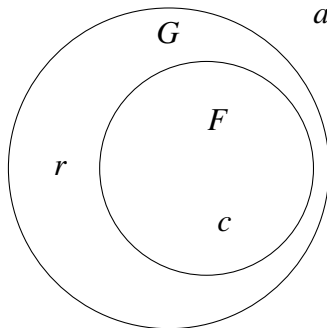
We can also get an idea of how the world is set up by drawing arrows to represent 2-place relations. For instance, the picture below represents the rest of the interpretation above. If there is an arrow connecting u and v, this represents "Ruv" holding in our interpretation. If there is no arrow from u to v, then "Ruv" does not hold.
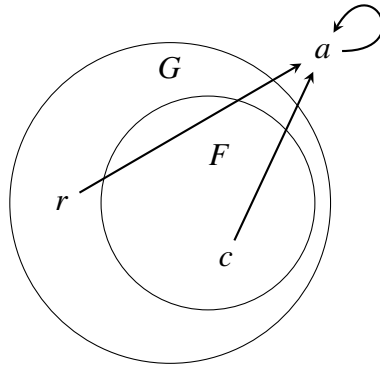
The nice thing about such diagrams is that if you draw it correctly, you don't have to think too hard about whether or not a schema is true or false on an interpretation: you can just *look* at the diagram. For instance, to determine if "Rxy" is true, since $\text{var}(x) = r$ and $\text{var}(y) = a$, you just determine if there is an arrow from r to a (which there is).

The downside is that there's not an easy way to generalize this to 3-place predicates, or anything more complicated than 3-place predicates for that matter. Also, if you have more than one binary relation, you'll need to distinguish differents arrows (maybe have one dotted, one dashed, etc.), which can get messy. But for you homework, you'll only be dealing with one binary relation at a time.

EXAMPLE 3
Which schemata are satisfied in which arrow diagram below?
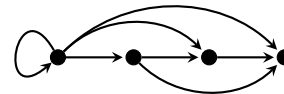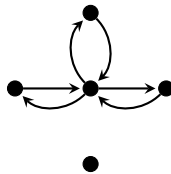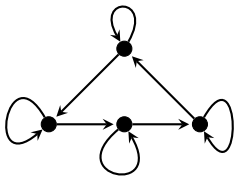
(i) $\forall x\, Rxx$

(ii) $\forall x\, \exists y\, Rxy$

(iii) $\exists x\, \forall y\, Rxy$

(iv) $\forall x\, \forall y\, (Rxy \supset Ryx)$

(v) $\forall x\, \forall y\, \forall z\, (Rxy \,.\, Ryz \supset Rxz)$

(vi) $\exists x\, \forall y\, -Ryx$

We can translate each of these schemata into "arrow-speak." Below are their translations. You should try to translate them yourself before reading my version.

(i)   "Every point has an arrow from itself to itself."

(ii)   "Every point has an arrow from itself."

(iii) "Some point has an arrow to every point."

(iv)   "Every arrow has a reverse arrow."

(v)   "If a point is connected to another via two arrows, then it's connected via one."
       Put another way: "If there's a 2-step path from a to b, then there's a 1-step path."

(vi)   "Some point has no arrow pointing to it."

 (Notice that (vi) is *not* just the negation of (ii). Do you see why?)

EXAMPLE 4   (CONT.)
So which schemata are true in which diagram? You should check yourself before reading my answers...

(i)   First diagram: ⊤   (iii) First diagram: ⊥   (v)   First diagram: ⊥
       Second diagram: ⊥   Second diagram: ⊥   Second diagram: ⊥
       Third diagram: ⊥   Third diagram: ⊤   Third diagram: ⊤

(ii)   First diagram: ⊤   (iv)   First diagram: ⊥   (vi)   First diagram: ⊥
       Second diagram: ⊥   Second diagram: ⊤   Second diagram: ⊤
       Third diagram: ⊥   Third diagram: ⊥   Third diagram: ⊤

    Note: if we had removed the arrow from the far left point to the far right point in the third diagram, (v) would *not* hold any more.

EXERCISE 5
Give an interpretation in which all three of these schemata are true.

(i) $\forall x - Rxx$

(ii) $\forall x \, \exists y \, Rxy$

(iii) $\forall x \, \forall y \, \forall z \, (Rxy \, . \, Ryz \supset Rxz)$

(*Hint*: How many points does it take to make these true? It's not 1, it's not 2, it's not 3,...)