Logical Consequence: Syntax

There are (at least) two parts of any language:

- *Syntax*: grammatical rules telling us what counts as a well-formed sentence.
- *Semantics*: assignment of meaning or truth value to well-formed sentences.

Similarly, there are two kinds of logical consequence. So far, we've focused on the *semantic* kind of logical consequence: A implies B iff every *interpretation* making A true also makes B true. Now we look at the *syntactic* kind: A implies B iff we can infer B from A's *logical form*. To figure out how we can do that, we must learn certain *rules of inferences* which all say something like: "If you've already obtained a schema of the form φ, you may infer a schema of the form ψ."

Deduction systems come in many different flavors. Below, we'll examine Goldfarb's deduction system to see how it works. However, I should note that there are plenty of other deduction systems out there which are equally as good for our purposes (the one I particularly like is the *Fitch* system, which involves these vertical lines I sometimes draw; you can google it or ask me).

1 Independent vs. Dependent Premises

When asked to anaylze an argument in previous homeworks, the arguments usually look something like this:

(1) No faculty member who plays the harmonica is brave.

(2) No one who isn't brave will survive the zombie apocalypse.

(3) Every philosophy faculty member plays the harmonica.

(C) No philosophy faculty member will surive the zombie apocalypse.

To figure out whether or not the premises imply the conclusion, we had to argue about what things are in the extension of predicates like "① is a faculty member", "① will survive the zombie apocalypse", etc. What does the argument look like in full though? How do we fill in the steps to get from the premises to the conclusion. Well, here's one way:

(1) No faculty member who plays the harmonica is brave. (**Premise**)

(2) No one who isn't brave will survive the zombie apocalypse. (**Premise**)

(3) Every philosophy faculty member plays the harmonica. (**Premise**)

(4) Hence, every philosophy faculty member is a faculty member who plays the harmonica. (**Follows from (3)**)

(5) So every philosophy faculty member is not brave, i.e. no philosophy faculty member is brave. (**Follows from (1) and (4)**)

(C) Therefore, no philosophy faculty member will surive the zombie apocalypse. (**Follows from (2) and (5)**)

Lines (1), (2), and (3) are **independent premises**. We didn't infer them from anything; we're just assuming they're true for the sake of argument. By contrast, lines (4) and (5) are **dependent premises**. We're not just assuming they're true; rather, we inferred them from the independent premises and from statements we've previously inferred.

Goldfarb's deduction system requires that you keep track of *both* independent and dependent premises. Consider the argument above. Line (4) followed only from line (3), so line (4) only depends on line (3); that is, the truth of line (4) only depends on the truth of line (3). By constrast, line (5) followed from lines (1) and (4), and since line (4) depends on line (3), line (5) depends on *both* lines (1) and (3); that is, the truth of (5) depends on lines (1) and (3).

2 TF Deductions

Each line in a deduction must cite three things: it must cite (a) the independent premises that line depends on, (b) the dependent premises that line depends on, and (c) the *rule of inference* that was invoked in inferring that line. The independent premises are cited in square brackets to the left of the line number. The dependent premises are cited to the right of the schema, along side the rule of inference.

Consider this "mock" deduction done in TF-logic alone:

[1]	(1)	$p \supset q$	P
[2]	(2)	$q \supset r$	P
[3]	(3)	p	P
[1, 3]	(4)	q	*Modus Ponens*: (1), (3)
[1, 2, 3]	(5)	r	*Modus Ponens*: (2), (4)

Again, the square brackets indicate which of the independent premises is invoked at that line. The next number is just a line number. Then after the schema on that line, we cite the rule of inference we used to infer that line, as well as which line numbers the rule of inference invoked.

The first three lines list the independent premises we'll make use of in the argument. Their "rule of inference" is denoted simply as "P" for "Premise". The square brackets to the left of their line number must be the same as their line number (for convenience). Usually you put all of your premises at the top of the deduction, but you don't have to; whatever's easiest.

Line (4) is the result of applying the *modus ponens* rule to "$p \supset q$" and "p". Hence, to the right of line (4), I've made note of this fact. To the left, I've made a note of which of my initial premises I've invoked in inferring "q", which are just lines (1) and (3).

Line (5) is also the result of applying *modus ponens*, but this time to "$q \supset r$", which is an independent premise, and "q" which is a *derived* or *dependent* premise that I got above in line (4). Hence, to the right of line (5), I've noted which line numbers were involved in the use of *modus ponens*, which are just lines (2) and (4).

But now to the left of line (5), I need to reference which of the *independent* premises I've used. To figure out which these are, I just need to go to line (2) and line (4) (which were invoked in my use of *modus ponens*), determine the independent premises of *those* lines, and then combine them. Line (2) is an independent premise, so it only depends on premise (2). On the other hand, line (4) is a dependent premise that depends on premises (1) and (3). So line (5) depends on (1), (2), and (3).

Here's another example (still in TF-logic):

$[1]$	(1)	$-p \vee q$	P
$[2]$	(2)	$q \supset r$	P
$[3]$	(3)	$-r$	P
$[2,3]$	(4)	$-q$	*Modus Tollens*: (2), (3)
$[1,2,3]$	(5)	$-p$	*Disjunctive Syllogism*: (1), (4)
$[1,2]$	(6)	$-r \supset -p$	*Discharge* [3]: (5)

The rule of *discharge* here is an important rule: it allows you to "discharge" assumptions you've made, so long as you add those assumptions as antecedents of a conditional. This is basically what you're doing in conditional proofs. Conditional proofs are of the form: "Suppose A. Then ... Therefore, B. Hence, if A then B." A is just a supposition for the sake of argument; once you conclude B from A, you can infer "If A then B" without keeping your supposition that A is true in addition.

Note that with the rule of *discharge*, you could potentially discharge all of your independent premises. For instance, I could keep using discharge on the argument above to get:

[1]	(7)	$(q \supset r) \supset (-r \supset -p)$	*Discharge* [2]: (6)
	(8)	$(-p \vee q) \supset ((q \supset r) \supset (-r \supset -p))$	*Discharge* [1]: (7)

where the conclusion doesn't cite any independent premise. Any line in your deduction which cites no independent premises is a *valid* schema. In fact, you can read line (5) (for instnace) as saying "Premises (1), (2), and (3) together imply (5)". So each line in your deduction is an implication, and you're just chaining implications together in a systematic fashion to show that other implications hold.

When quantifiers get involved, things get slightly messy, but the same general idea applies. In the next section, I list all of the rules in Goldfarb's deduction system. He has abbreviations for each rule, which make it easier to cite (but you may want to memorize the abbreviations so you don't have to keep referring to them).

(By the way, the rules below are all super-wordy. I would suggest first looking at some examples of deductions, and then coming back and reading the rules in more detail.)

3 Rules of Inference

Rule P (Premise Introduction): On any line (n), you may add any schema as an independent premise. To the left, you must put [n]. To the right, you simply invoke "P".

Rule D (Discharge): If schema A is on line (n), and schema B is on line (m), you may infer $A \supset B$ after line (m). To the left, you must cite all of the independent premises used on line (m) except (n). To the right, you must invoke "D" and cite both [n] and line (m).

Rule TF (TF-Implication): If schemata $A_1, \ldots, A_k$ are on lines (n_1), ..., (n_k) respectively, and $A_1, \ldots, A_k$ TF-imply B, you may infer B. To the left, you must cite all of the independent premises of (n_1), ..., (n_k). To the right, you must invoke "TF" and cite lines (n_1), ..., (n_k).

Rule CQ (Conversion of Quantifiers): If quantified schema is on line (n), you may infer any "conversion" of that schema, i.e. you may push/pull negations out (whilst flipping quantifiers) to get the equivalent schema. To the left, you must cite the independent premises of line (n). To the right, you must invoke "CQ" and cite line (n).

Rule UI (Universal Instantiation): If schema $\forall v\ A$ is on line (n), you may infer any instance of $\forall v\ A$. To the left, you must cite the independent premises of line (n). To the right, you must invoke "UI" and cite line (n).

Rule UG (Universal Generalization): If schema A is on line (n), and the variable v does not occur free in any of the independent premises cites to the left of line (n), you may infer $\forall v\ A$. To the left, you must cite the independent premises of line (n). To the right, you must invoke "UG" and cite line (n).

Rule EG (Existential Generalization): If schema A is on line (n), and is an instance of an existential schema, you may infer the existential. To the left, you must cite the independent premises of line (n). To the right, you must invoke "EG" and cite line (n).

Rule EII (Existential Instantiation Introduction): If schema $\exists v\ A$ is on line (n), you may infer any instance of $\exists v\ A$ provided that the instantial variable does not occur free in any line above (and including) (n). To the left, you must cite the independent premises of line (n) *in addition to* [n]. To the right, you must invoke "EII" and cite both line (n) and the instantial variable.

Rule EIE (Existential Instantiation Elimination): If schema A is on line (n), and line (n) depends on a premise (j) where Rule EII was invoked, *and* the instantial variable of line (j) does not occur free in the schema on line (n), you may infer A. To the left, you must cite the independent premises of line (n) except for (j). To the right, you must invoke "EIE" and cite both [j] and line (n).

4 Examples of Deductions: Universals

EXAMPLE 1
Use a deduction to prove: "$\forall x\ (Fx \supset Gx)$" and "$\forall x\ (Gx \supset Hx)$" together imply "$\forall x\ (Fx \supset Hx)$". (Aristotle anyone?)

Your independent premises are "$\forall x\ (Fx \supset Gx)$" and "$\forall x\ (Gx \supset Hx)$". So your first two lines should be Premise Introduction (Rule P) with these two schemata.

[1]	(1)	$\forall x(Fx \supset Gx)$	P
[2]	(2)	$\forall x(Gx \supset Hx)$	P

Usually, if you have universal schemata as premises, you'll need to instantiate them, i.e. use Rule UI, which says you may infer from a universal schema its instances. Hence, the next lines should be:

[1]	(3)	$Fx \supset Gx$	UI: (1)
[2]	(4)	$Gx \supset Hx$	UI: (2)

We can use any instantial variable (so long as the result is an instance). We usually just choose the same variable for convenience (except when using EII: there, you *must* use a different variable).

But, by TF-logic, "$p \supset q$" and "$q \supset r$" imply "$p \supset r$". So:

[1, 2]	(5)	$Fx \supset Hx$	TF: (3), (4)

Note that we had to combine the independent premises of lines (3) and (4). Finally, according to Rule UG, if x does not occur free in any of the independent premises this line depends on (viz. (1) and (2)), we can bind x with a universal quantifier. Hence our conclusion:

[1, 2]	(6)	$\forall x(Fx \supset Hx)$	UG: (5)

If this last step didn't make sense, think of it this way: in English, the argument would go, "Suppose every F is G and every G is H. Suppose Jim (say) is an F. Then by our first supposition, Jim is a G. But then by our second supposition, Jim is an H. Hence, if Jim is an F, then he is an H. But there's nothing special about Jim; he was just some arbitrary F. Hence, *every* F is H." In our deduction, free x is acting like a name of an arbitrary object, like the name "Jim" above, and we introduce this name when we use UI. But once we conclude what we want about free x, since it didn't matter what x was (it could have been *any* object), it follows that what we concluded about x holds of *any* object, which is what we conclude when we use UG. To put it loosely: UI allows you to introduce "temporary names", while UG takes them back.

When all is said and done, your deduction should look as follows:

[1]	(1)	$\forall x(Fx \supset Gx)$	P
[2]	(2)	$\forall x(Gx \supset Hx)$	P
[1]	(3)	$Fx \supset Gx$	UI: (1)
[2]	(4)	$Gx \supset Hx$	UI: (2)
[1, 2]	(5)	$Fx \supset Hx$	TF: (3), (4)
[1, 2]	(6)	$\forall x(Fx \supset Hx)$	UG: (5)

Your last line says "Premises (1) and (2) imply "$\forall x\ (Fx \supset Hx)$" ", which is what you wanted.

EXAMPLE 2

Use a deduction to show: "$\forall x\ (Fx \supset Gx)$" implies "$\forall x\ Fx \supset \forall x\ Gx$". (But of course, the converse doesn't hold!)

Again, your premise is "$\forall x\ (Fx \supset Gx)$", so you should start with Rule P.

[1]	(1)	$\forall x(Fx \supset Gx)$	P

Chances are, you'll have to instantiate this universal. But before we do that, note: the conclusion we want to draw has the form of a conditional. To prove a conditional, we need to assume the antecedent, and then derive the consequent. So we'll need to add the antecedent of the conclusion as an extra premise (to be discharged later).

[2]	(2)	$\forall x Fx$	P

Now, since our premises are universals, we should probably instantiate them. And by TF-logic, we get:

[1]	(3)	$Fx \supset Gx$	UI: (1)
[2]	(4)	Fx	UI: (2)
[1, 2]	(5)	Gx	TF: (3), (4)

And again, remember we had to combine the independent premises of (3) and (4) at line (5). Finally, since x does not occur free in either of the premises, we can use UG:

[1, 2]	(6)	$\forall x Gx$	UG: (5)
[1]	(7)	$\forall x Fx \supset \forall x\, Gx$	D[2]: (6)

And as we did above, we can *discharge* Premise (2) to get the conditional we want on line

(7). All-in-all, the argument should look like:

[1]	(1)	$\forall x(Fx \supset Gx)$	P
[2]	(2)	$\forall x Fx$	P
[1]	(3)	$Fx \supset Gx$	UI: (1)
[2]	(4)	Fx	UI: (2)
[1, 2]	(5)	Gx	TF: (3), (4)
[1, 2]	(6)	$\forall x Gx$	UG: (5)
[1]	(7)	$\forall x Fx \supset \forall x\, Gx$	D[2]: (6)

Again, the use of *discharge* is like conditional proof: "Working under the assumption of Premise (1), suppose "$\forall x\ Fx$" is true. Then (via some inferences) "$\forall x\ Gx$" is true. Hence, still working under the assumption of Premise (1), "$\forall x\ Fx \supset \forall x\ Gx$" is true."

WARNING: This is a *bad* deduction:

[1]	(1)	$\forall x Fx \supset Gy$	P
[2]	(2)	$\forall x Fx$	P
[1, 2]	(3)	Gy	TF: (1), (2)
[1, 2]	(4)	$\forall y Gy$	UG: (3) (!!DON'TDOITIT'SATRAP!!)
[1]	(5)	$\forall x Fx \supset \forall y\, Gy$	D[2]: (4)

In English, this would amount to a proof that "Mary is a girl if everyone is human" implies "If everyone is human, then everyone is a girl." The problem with this deduction (if it wasn't clear from the above) is that our use of UG is not allowed: according to rules given above, we can only use Rule UG if the variable we're binding doesn't occur free in any of our independent premises. And this isn't true of Premise (1): y *does* occur free in (1). So we can't make this inference (which is good, since the inference is invalid).

What the above illustrates is that you must pay careful attention to the detail of the rules of inference presented above. Conditions like those in UG, EII, and EIE cannot be ignored!

5 Examples of Deduction: Existential

Existentials are trickier to deal with in deductions. Their generalization rule is okay, but their instantiation rule is a bit contrived in this deduction system. Careful attention to detail is advised for these examples.

Example 3
Use a deduction to prove: "$\exists x\ (Fx\ .\ Gx)$" implies "$\exists x\ Fx$".

This should be easy to see is true, but we want to see what it looks like as a deduction. As always, start with your premise using Rule P:

$$[1] \quad (1) \qquad \exists x(Fx\ .\ Gx) \qquad \text{P}$$

Usually, if you have an existential premise, you'll probably have to instantiate it. Instantiation for existentials comes in two parts: introduction of an instantial variable and elimination of our instantial assumption. Let's see how it works in deductions first, and then I'll try to elaborate in English what's going on.

So, to instantiate an existential, we first introduce an instantial variable:

$$[1,2] \quad (2) \qquad Fx\ .\ Gx \qquad \text{EII}(x)\text{: } (1)$$

Notice we've indicate the instantial variable in our rule of inference, viz. free x. Also note we've added this same line *as an independent premise*. The reason is that this "independent premise" will eventually be *forced* to discharge. Now, via TF-logic:

$$[1,2] \quad (3) \qquad Fx \qquad \text{TF: } (2)$$

So by Existential Generalization:

$$[1,2] \quad (4) \qquad \exists x Fx \qquad \text{EG: } (3)$$

Finally, we need to eliminate our instantial assumption. That is, since we inferred "$\exists x\ Fx$" from "$Fx\ .\ Gx$," the conclusion still holds *so long as* "$\exists x\ (Fx\ .\ Gx)$" is true. So:

$$[1] \quad (5) \qquad \exists x Fx \qquad \text{EIE}[2]\text{: } (4)$$

Notice that now our independent premises have reduced back down to just Premise (1): we've effectively *discharged* our second "premise".

So what exactly is going on? The idea is that we're reasoning in the following way: "Suppose there is an F that's G. Let's call such an F G "Jim". Since Jim is F and G, he is F. Hence, there exists an F, assuming Jim is an F and a G. But our choice of "Jim" doesn't matter, since we know there is at least one F that's G, and any old F G would do. Hence, there exists an F, so long as we're still assuming there exists an F that's G."

This English argument is exactly what we're doing in the deduction. When we say, "Let's call such an F G "Jim"," we're using EII by introducing a new name for an object which acts as a "witness" to our existential, i.e. an object that is both F and G. We then make some inferences from the fact that our Jim is F and G, and we conclude that there exists at least one F. So if we assume Jim is F and G, then there is an F. But our choice of "Jim" wasn't important: the only thing that was important was that there was some F that was G (which was our whole reason for introducing "Jim" in the first place). So we can discharge the assumption that Jim is F and G, which is what we do when we use EIE in line (5).

Our final deduction is below. You should review it with the above comments in mind to see if the uses of EII and EIE make more sense. It may take a while to sink in, so you may just need to reread the above comments again until it does.

$[1]$	(1)	$\exists x(Fx \,.\, Gx)$	P
$[1,2]$	(2)	$Fx \,.\, Gx$	EII(x): (1)
$[1,2]$	(3)	Fx	TF: (2)
$[1,2]$	(4)	$\exists x Fx$	EG: (3)
$[1]$	(5)	$\exists x Fx$	EIE[2]: (4)

We will *always* need to use EIE if we've used EII earlier in our deduction. Because of this, instead of repeating the same schema twice, we can combine two lines into one as follows:

$[1]$	(1)	$\exists x(Fx \,.\, Gx)$	P
$[1,2]$	(2)	$Fx \,.\, Gx$	EII(x): (1)
$[1,2]$	(3)	Fx	TF: (2)
$[1,\not{2}]$	(4)	$\exists x Fx$	EG: (3); EIE[2]

where we cross out the 2 and put both rules used as above.

EXAMPLE 4

Use a deduction to prove: "$\forall x\,(Gx \supset Fx)$" and "$\exists x\,(Gx \,.\, Hx)$" together imply "$\exists x\,(Fx \,.\, Hx)$".

As always, two premises, two uses of Rule P:

$[1]$	(1)	$\forall x(Gx \supset Fx)$	P
$[2]$	(2)	$\exists x(Gx \,.\, Hx)$	P

Chances are, you'll need to instantiate both of these. Does it matter which one we instantiate first? Since we're not dealing with embedded quantifiers, the answer is no. (If you were dealing with embedded quantifiers, you'd have to go in order working your way inward, but that's not surprising.) However, in this case, it's more natural to do existential instantiation first. You could do it in the other order, but you'd get this result:

$[1]$	(3)	$Gx \supset Fx$	UI: (1)
$[2,4]$	(4)	$Gx \,.\, Hx$	EII(x): (2)

which just looks a little weird, having no Premise (3) to refer to (unfortunately, we have to do it this way when quantifiers are embedded). So I'll do it the other way:

[2, 3]	(3)	$Gx \,.\, Hx$	EII(x): (2)
[1]	(4)	$Gx \supset Fx$	UI: (1)

Via TF-logic:

[2, 3]	(5)	Gx	TF: (3)
[1, 2, 3]	(6)	Fx	TF: (3), (5)
[1, 2, 3]	(7)	$Fx \,.\, Hx$	TF: (3), (6)

You could combine all of these steps into one step, but I've expanded it out so it's easier to see the thought process. So now, using EG and discharging our instantial assumption:

[1, 2, 3]	(8)	$\exists x(Fx \,.\, Hx)$	EG: (7)
[1, 2]	(9)	$\exists x(Fx \,.\, Hx)$	EIE[3]: (8)

Put all together:

[1]	(1)	$\forall x(Gx \supset Fx)$	P
[2]	(2)	$\exists x(Gx \,.\, Hx)$	P
[2, 3]	(3)	$Gx \,.\, Hx$	EII(x): (2)
[1]	(4)	$Gx \supset Fx$	UI: (1)
[2, 3]	(5)	Gx	TF: (3)
[1, 2, 3]	(6)	Fx	TF: (3), (5)
[1, 2, 3]	(7)	$Fx \,.\, Hx$	TF: (3), (6)
[1, 2, 3]	(8)	$\exists x(Fx \,.\, Hx)$	EG: (7)
[1, 2]	(9)	$\exists x(Fx \,.\, Hx)$	EIE[3]: (8)

6 Examples of Deduction: Polyadic

All of the examples above were done in monadic logic. Polyadic deductions are exactly as you'd expect, except now we have embedded quantifiers. There aren't any hidden trapdoors here; we just proceed as before.

EXAMPLE 5
Use a deduction to prove: "$\exists x\ \forall y\ Gxy$" implies "$\exists x\ Gxx$".

[1]	(1)	$\exists x \forall y\ Gxy$	P
[1]	(2)	$\forall y Gxy$	EII(x): (1)
[1, 3]	(3)	Gxx	UI: (2)
$[1, \not{3}]$	(4)	$\exists x Gxx$	EG: (3); EIE[3]: (4)

EXAMPLE 6
Use a deduction to prove: (i) "$\forall x\ \forall y\ \ (\exists z\ (Txz\ .\ Ayz) \supset Ryz)$" and (ii) "$\forall x\ \forall y\ \ (Lx \supset Ayx)$" together imply "$\forall x\ \ (\exists z\ (Lz\ .\ Txy) \supset \forall z\ Rzx)$". (Fill in the blanks yourself, including the rules of inference.)

[1]	(1)	$\forall x \forall y\ \ (\exists z\ (Txz\ .\ Ayz) \supset Ryz)$	P
[2]	(2)	$\forall x \forall y\ \ (Lx \supset Ayx)$	P
[3]	(3)	$\exists y\ (Ly\ .\ Txy)$	P
[3, 4]	(4)	$Lu\ .\ Txu$	
[ ]	(5)	$\forall y (Lu \supset Ayu)$	
[ ]	(6)		UI: (5)
[ ]	(7)	$Txu\ .\ Ayu$	
[ ]	(8)		EG: (7); EIE[4]
[1]	(9)		UI: (1)
[1]	(10)		UI: (9)
[ ]	(11)	Ryx	
[ ]	(12)	$\forall z Rzx$	
[ ]	(13)		D[3]: (12)
[ ]	(14)	$\forall x\ (\exists y\ (Ly\ .\ Txy) \supset \forall z\ Rzx)$	