Dynamic Epistemic Logic

Phil 143 Worksheet

1. Alice and Bert are told that Cheryl's birthday is among the following list:

- May 15, 16, 19
- June 17, 18
- July 14, 16
- August 14, 15, 17

Cheryl tells Alice the month of her birthday and Bert the day. Then the following announcements are made:

A: I don't know her birthday, but I know Bert doesn't know either.

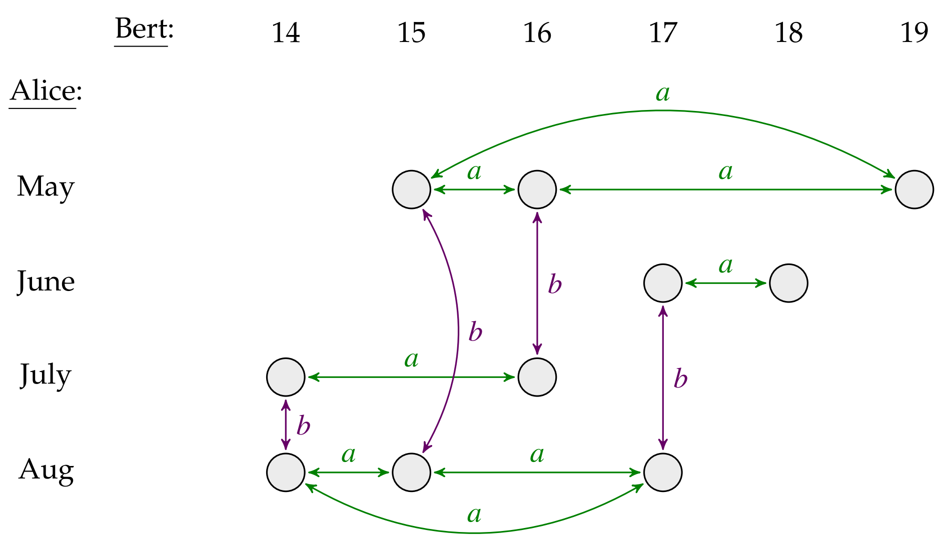
B: Now I know her birthday.

A: Now I know her birthday.

Draw an epistemic model which represents this situation, and draw the various updated models after each announcement. Use this to determine Cheryl's birthday.

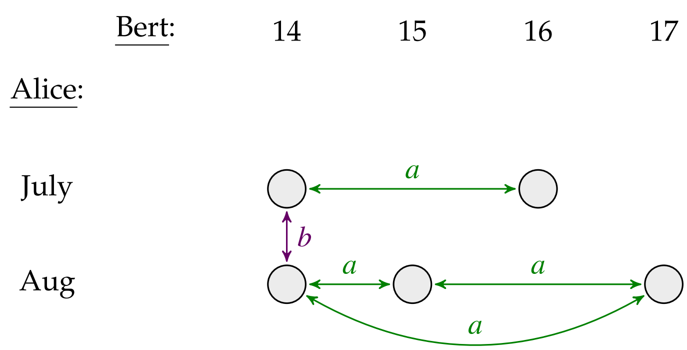
This problem went viral this week: click here a statement of the problem from the NY Times.

▶ Solution: Here's the initial model (with reflexive arrows left out):

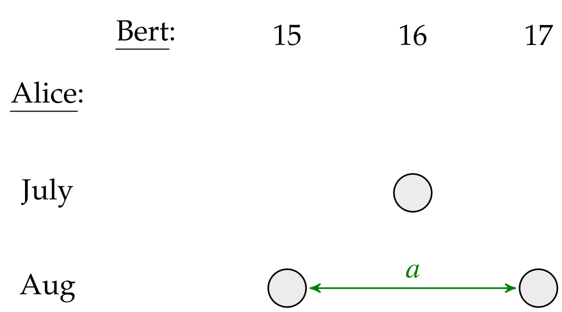


Now, once Alice says that she knows Bert doesn't know, we need to eliminate all the worlds where Alice doesn't know that Bert doesn't know. That is, we must eliminate all the worlds where Alice can't rule out a world where Bert knows the birthday.

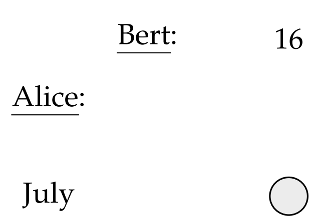
Since Bert knows the birthday if it's May 19^{th} or if it's June 18^{th}, that means that we must eliminate all the worlds where Alice can't rule out May 19^{th}, and all the worlds where Alice can't rule out June 18^{th}. But the worlds where Alice can't rule out May 19^{th} are exactly the May worlds, and the worlds where Alice can't rule out June 19^{th} are exactly the June worlds. So we must remove all of the May and June worlds, leaving:



Now Bert says he knows the date. So we must eliminate all the remaining worlds where Bert doesn't know the date. The only worlds where he doesn't know the date are dates with the 14^{th}. So we're left with:



And now when Alice says she knows, we must eliminate the worlds where Alice don't know, leaving just Cheryl's birthday:



2. For each formula below, determine whether the formula is valid in public announcement logic. If it is, prove it. If not, provide a counterexample.

(a) $[!\alpha]\,[!\beta]\,\varphi \leftrightarrow [!(\alpha \wedge [!\alpha]\beta)]\,\varphi$

► Solution: Valid. We know that

$$\begin{aligned}\mathcal{M}, w \models [!\alpha]\,[!\beta]\,\varphi &\Leftrightarrow \text{if } \mathcal{M}, w \models \varphi, \text{then} \mathcal{M}_\alpha, w \models [!\beta]\,\varphi \\ &\Leftrightarrow \text{if } \mathcal{M}, w \models \varphi, \text{then if } \mathcal{M}_\alpha, w \models \beta, \text{then } \mathcal{M}_{\alpha,\beta}, w \models \varphi \\ &\Leftrightarrow \text{if } \mathcal{M}, w \models \alpha \text{ and } \mathcal{M}_\alpha, w \models \beta, \text{then } \underline{\mathcal{M}_{\alpha,\beta}, w \models \varphi}.\end{aligned}$$

And we know that

$$\begin{aligned}\mathcal{M}, w \models [!(\alpha \wedge [!\alpha]\beta)]\,\varphi \quad &\Leftrightarrow \quad \text{if } \mathcal{M}, w \models \alpha \wedge [!\alpha]\beta, \text{then } \mathcal{M}_{\alpha \wedge [!\alpha]\beta}, w \models \varphi \\ &\Leftrightarrow \quad \text{if } \mathcal{M}, w \models \alpha \text{ and } \mathcal{M}, w \models [!\alpha]\beta, \\ &\qquad \text{then } \mathcal{M}_{\alpha \wedge [!\alpha]\beta}, w \models \varphi \\ &\Leftrightarrow \quad \text{if } \mathcal{M}, w \models \alpha \text{ and } \mathcal{M}_\alpha, w \models \beta, \\ &\qquad \text{then } \underline{\mathcal{M}_{\alpha \wedge [!\alpha]\beta}, w \models \varphi}.\end{aligned}$$

Notice that the only difference between the two clauses is the underlined part. Thus, it suffices to show that the underlined parts are equivalent, i.e., that:

$$\mathcal{M}_{\alpha,\beta}, w \models \varphi \quad \Leftrightarrow \quad \mathcal{M}_{\alpha \wedge [!\alpha]\beta}, w \models \varphi.$$

In fact, we'll show that $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha \wedge [!\alpha]\beta}$ are the same model. Here's intuitively why. $\mathcal{M}_{\alpha,\beta}$ contains the worlds where α was true in $\mathcal{M}$, and such that β was true in $\mathcal{M}_\alpha$. $\mathcal{M}_{\alpha \wedge [!\alpha]\beta}$ contains the worlds where $\alpha \wedge [!\alpha]\beta$ was true in $\mathcal{M}$. But those are the worlds where α was true in $\mathcal{M}$, and where β was true $\mathcal{M}_\alpha$. So they consist of the same worlds, and hence must be the same submodel of $\mathcal{M}$.

Here's a more formal explanation: let $v \in W$. Then:

$$\begin{aligned}v \in W_{\alpha,\beta} \quad &\Leftrightarrow \quad \mathcal{M}, v \models \alpha \text{ and } \mathcal{M}_\alpha, v \models \beta \\ &\Leftrightarrow \quad \mathcal{M}, v \models \alpha \text{ and } \mathcal{M}, v \models [!\alpha]\beta \\ &\Leftrightarrow \quad \mathcal{M}, v \models \alpha \wedge [!\alpha]\beta \\ &\Leftrightarrow \quad v \in W_{\alpha \wedge [!\alpha]\beta}.\end{aligned}$$

So $W_{\alpha,\beta} = W_{\alpha \wedge [!\alpha]\beta}$. But then $R_{\alpha,\beta} = R \cap W_{\alpha,\beta} \times W_{\alpha,\beta} = R \cap W_{\alpha \wedge [!\alpha]\beta} \times W_{\alpha \wedge [!\alpha]\beta} = R_{\alpha \wedge [!\alpha]\beta}$. Similarly, $V_{\alpha,\beta}(p) = V(p) \cap W_{\alpha,\beta} = V(p) \cap W_{\alpha \wedge [!\alpha]\beta} = V_{\alpha \wedge [!\alpha]\beta}(p)$. So the set of worlds, the accessibility relations, and the valuation function from each model are all the same. So they're the same model. Hence, we've shown that $\mathcal{M}_{\alpha,\beta}, w \models \varphi \Leftrightarrow \mathcal{M}_{\alpha \wedge [!\alpha]\beta}, w \models \varphi$, as desired.

(b) $[!\alpha]\,\mathsf{K}_a\varphi \leftrightarrow \mathsf{K}_a\,[!\alpha]\,\varphi$

▶ Solution: Not valid. The lefthand side amounts to:

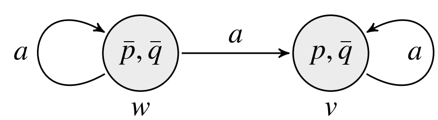
$$\begin{aligned}\mathcal{M}, w \models [!\alpha]\,\mathsf{K}_a\varphi \quad &\Leftrightarrow \quad \text{if } \mathcal{M}, w \models \alpha, \text{ then } \mathcal{M}_\alpha, w \models \mathsf{K}_a\varphi\\ &\Leftrightarrow \quad \text{if } \mathcal{M}, w \models \alpha, \text{ then } \forall v \in W_\alpha: \text{ if } wR_{a,\alpha}v, \text{ then } \mathcal{M}_\alpha, v \models \varphi.\end{aligned}$$

By contrast, the righthand side amounts to:

$$\begin{aligned}\mathcal{M}, w \models \mathsf{K}_a\,[!\alpha]\,\varphi \quad &\Leftrightarrow \quad \forall v \in W: \text{ if } wR_av, \text{ then } \mathcal{M}, v \models [!\alpha]\,\varphi\\ &\Leftrightarrow \quad \forall v \in W: \text{ if } wR_av, \text{ then if } \mathcal{M}, v \models \alpha, \text{ then } \mathcal{M}_\alpha, v \models \varphi.\end{aligned}$$

Notice that the first is vacuously true when $\mathcal{M}, w \not\models \alpha$. By contrast, the latter is not obviously trueh when $\mathcal{M}, w \not\models \alpha$. This suggests that the two formulas aren't equivalent.

Indeed, consider the model below.



Consider $[!p]\,\mathsf{K}_a q \to \mathsf{K}_a\,[!p]\,q$. Notice that $\mathcal{M}, w \models [!p]\,\mathsf{K}_a q$ vacuously because $\mathcal{M}, w \not\models p$. However, $\mathcal{M}, w \not\models \mathsf{K}_a\,[!p]\,q$ because there's a world, namely v, such that wR_av but $\mathcal{M}, v \not\models [!p]\,q$.

You can show that, actually, $[!\alpha]\,\mathsf{K}_a\varphi \leftrightarrow (\alpha \to \mathsf{K}_a\,[!\alpha]\,\varphi)$. You should try to prove this as an exercise.

(c) $\langle !\alpha\rangle\,\varphi \to [!\alpha]\,\varphi$

▶ Solution: Valid. Suppose $\mathcal{M}, w \models \langle !\alpha\rangle\varphi$. That means $\mathcal{M}, w \models \alpha$ and $\mathcal{M}_\alpha, w \models \varphi$. But then vacuously if $\mathcal{M}, w \models \alpha$, then $\mathcal{M}_\alpha, w \models \varphi$. So $\mathcal{M}, w\,[!\alpha]\,\varphi$.

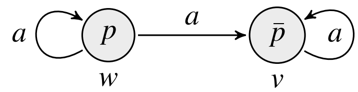
Notice this means that if we defined a relation $R_{!\alpha}$, then for all $w, v, v' \in W$, if $wR_{!\alpha}v$ and $wR_{!\alpha}v'$, then $v = v'$, i.e., $R_{!\alpha}$ would be ***partially functional***.

3. Which of the standard modal axioms **T**, **D**, **B**, **4**, **5** hold in public announcement logic?

▶ Solution: None of them are valid. I will simple present the models invalidating each axiom. I leave it up to you to show that this model works.

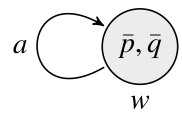
T: $[!\alpha]\,\varphi \to \varphi$.

Consider $[!p]\,\mathsf{K}_a p \to \mathsf{K}_a p$ in this model:



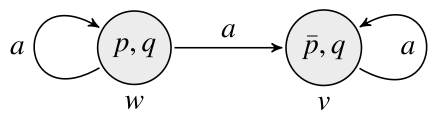
D: $[!\alpha]\,\varphi \to \langle !\alpha\rangle\,\varphi$.

Consider $[!p]\,q \to \langle !p\rangle\,q$ in this model:



B: $\varphi \to [!\alpha]\,\langle !\alpha\rangle\,\varphi$.

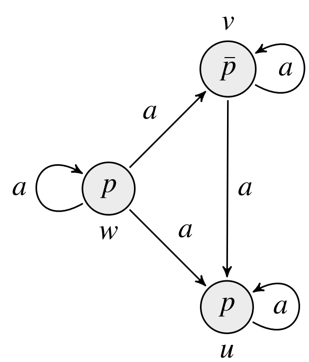
Consider $q \to [!(p \wedge \neg\,\mathsf{K}_a p)]\,\langle !(p \wedge \neg\,\mathsf{K}_a p)\rangle\,q$ in this model:



4: $[!\alpha]\,\varphi \to [!\alpha]\,[!\alpha]\,\varphi$.

Let $\mathsf{W}_a p$ abbreviate $\mathsf{K}_a p \vee \mathsf{K}_a \neg\, p$ (read $\mathsf{W}_a p$ as saying "agent a knows whether or not p is true").

Consider $[!\neg\,\mathsf{K}_a\mathsf{W}_a p]\,\neg\,\mathsf{K}_a p \;\to\; [!\neg\,\mathsf{K}_a\mathsf{W}_a p]\,[!\neg\,\mathsf{K}_a\mathsf{W}_a p]\,\neg\,\mathsf{K}_a p$ in this model:



5: $\langle !\alpha\rangle\,\varphi \to [!\alpha]\,\langle !\alpha\rangle\,\varphi$.

Consider $\langle !(p \wedge \neg\,\mathsf{K}_a p)\rangle\,q \;\to\; [!(p \wedge \neg\,\mathsf{K}_a p)]\,\langle !(p \wedge \neg\,\mathsf{K}_a p)\rangle\,q$ in this model: