Extensions of **K**

Phil 143 Worksheet

1. Produce axiomatic proofs that show the following:

(a) $\vdash_{\mathbf{KT5}} \varphi \to \Box\Diamond\varphi$

▶ Solution:

1.	$\Box\underline{\neg\varphi} \to \underline{\neg\varphi}$	(T axiom)
2.	$\varphi \to \neg\Box\neg\varphi$	(PL, 1)
3.	$\varphi \to \Diamond\varphi$	(Dual, 2)
4.	$\Diamond\varphi \to \Box\Diamond\varphi$	(5 axiom)
5.	$\varphi \to \Box\Diamond\varphi$	(PL, 3, 4)

(b) $\vdash_{\mathbf{KB4}} \Box(\Diamond\varphi \to \psi) \to \Box(\varphi \to \Box\psi)$

▶ Solution:

1.	$\Box\underline{(\Diamond\varphi \to \psi)} \to \Box\Box\underline{(\Diamond\varphi \to \psi)}$	(4 axiom)
2.	$\Box(\Diamond\varphi \to \psi) \to (\Box\Diamond\varphi \to \Box\psi)$	(K axiom)
3.	$\varphi \to \Box\Diamond\varphi$	(B axiom)
4.	$\Box(\Diamond\varphi \to \psi) \to (\varphi \to \Box\psi)$	(PL, 2, 3)
5.	$\Box\Box(\Diamond\varphi \to \psi) \to \Box(\varphi \to \Box\psi)$	(RK, 4)
6.	$\Box(\Diamond\varphi \to \psi) \to \Box(\varphi \to \Box\psi)$	(PL, 1, 5)

(c) $\vdash_{\mathbf{KD4}} \Box\Diamond\varphi \to \Diamond\varphi$

▶ Solution:

1.	$\Box\underline{\neg\varphi} \to \Box\Box\underline{\neg\varphi}$	(4 axiom)
2.	$\neg\Box\Box\neg\varphi \to \neg\Box\neg\varphi$	(PL, 1)
3.	$\neg\Box\neg\neg\Box\neg\varphi \to \neg\Box\neg\varphi$	(DN, 2)
4.	$\Diamond\Diamond\varphi \to \Diamond\varphi$	(Dual, 3)
5.	$\Box\underline{\Diamond\varphi} \to \Diamond\underline{\Diamond\varphi}$	(D axiom)
6.	$\Box\Diamond\varphi \to \Diamond\varphi$	(PL, 4, 5)

(d) $\vdash_{\mathbf{K5}} \Diamond\Diamond\varphi \to \Box\Diamond\varphi$

> ► Solution:
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> |---|---|---|
> | 1. | $\Diamond\varphi \to \Box\Diamond\varphi$ | (5 axiom) |
> | 2. | $\neg\Box\Diamond\varphi \to \neg\Diamond\varphi$ | (PL, 1) |
> | 3. | $\Box\neg\Box\Diamond\varphi \to \Box\neg\Diamond\varphi$ | (RK, 2) |
> | 4. | $\neg\Box\neg\Diamond\varphi \to \neg\Box\neg\Box\Diamond\varphi$ | (RK, 3) |
> | 5. | $\Diamond\Diamond\varphi \to \Diamond\Box\Diamond\varphi$ | (Dual, 4) |
> | 6. | $\Diamond\underline{\neg\Diamond\varphi} \to \Box\Diamond\underline{\neg\Diamond\varphi}$ | (5 axiom) |
> | 7. | $\neg\Box\Diamond\neg\Diamond\varphi \to \neg\Diamond\neg\Diamond\varphi$ | (PL, 6) |
> | 8. | $\neg\Box\neg\neg\Diamond\neg\Diamond\varphi \to \neg\Diamond\neg\Diamond\varphi$ | (DN, 7) |
> | 9. | $\Diamond\Box\Diamond\varphi \to \Box\Diamond\varphi$ | (Dual, 8) |
> | 10. | $\Diamond\Diamond\varphi \to \Box\Diamond\varphi$ | (PL, 5, 9) |

2. Let $\mathcal{F} = \langle W, R\rangle$ be a frame. Prove that $\Diamond p \to \Diamond\Diamond p$ is valid on $\mathcal{F}$ iff R is ***dense***: $\forall w, v \in W$ if wRv, then $\exists u \in W$ such that wRu and uRv.

> ► Solution: First, we'll show the left-to-right direction. Suppose $\Diamond p \to \Diamond\Diamond p$ is valid on $\mathcal{F}$. Let $w, v \in W$ be such that wRv. Define a model $\mathcal{M} = \langle W, R, V\rangle$ where $V(p) = \{v\}$. Then $\mathcal{M}, w \models \Diamond p$. But since $\Diamond p \to \Diamond\Diamond p$ is valid on $\mathcal{F}$, that means that $\mathcal{M}, w \models \Diamond\Diamond p$. So there must be a $u \in W$ such that wRu and $\mathcal{M}, u \models \Diamond p$. But by the definition of our V, v is the only world that satisfies p. So we must have uRv. Hence, for all $w, v \in W$, if wRv, then there is a u where wRu and uRv.
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> Next, we'll show the right-to-left direction. Suppose R is dense. Suppose $\mathcal{M} = \langle W, R, V\rangle$ for some valuation function V, and let $w \in W$ be such that $\mathcal{M}, w \models \Diamond p$. Then there is a $v \in W$ such that wRv and $\mathcal{M}, v \models p$. But by the fact that R is dense, there must be a $u \in W$ such that wRu and uRv. Hence, $\mathcal{M}, w \Diamond\Diamond p$. Since w was arbitrary, it follows that $\Diamond p \to \Diamond\Diamond p$ is valid on $\mathcal{F}$.

3. Find the frame correspondent for $\Diamond(p \to \Diamond p)$.

> ► Solution: It often helps to first suppose the formula is not valid, and then see where that leads. The negation of whatever first-order frame property you end up with will then be your answer.
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> Suppose $\Diamond(p \to \Diamond p)$ is not valid on $\mathcal{F}$. That means there's a model $\mathcal{M} = \langle W, R, V\rangle$ and a world $w \in W$ such that $\mathcal{M}, w \not\models \Diamond(p \to \Diamond p)$. So for all $v \in W$ such that wRv, $\mathcal{M}, v \not\models p \to \Diamond p$. That means for all $v \in W$ such that wRv, $\mathcal{M}, v \models p$ but $\mathcal{M}, v \not\models \Diamond p$. But this means that v cannot see any world that w can see. That is, this means that if wRv', then not-vRv'.

So what we ended up with was if $\Diamond(p \to \Diamond p)$ is not valid, then for some $w \in W$, we have that for all $v, v' \in W$ such that wRv and wRv', not-vRv'. In symbols, this looks like:

$$\exists w \in W \, \forall v, v' \in W: \; wRv \text{ and } wRv' \; \Rightarrow \; \text{not-}vRv'$$

The negation of this property is:

$$\boxed{\forall w \in W \;\; \exists v, v' \in W: \;\; wRv \;\; \text{and} \;\; wRv' \;\; \text{and} \;\; vRv'}$$

And this is our frame correspondent. What we just showed above was that if $\Diamond(p \to \Diamond p)$ is not valid on $\mathcal{F}$, then R doesn't have this property. To complete the proof, we must show the other direction.

Suppose R doesn't have this property, i.e., suppose there is a $w \in W$ such that for all $v \in W$, if wRv then not-vRv. Let $\mathcal{M} = \langle W, R, V \rangle$ where $V(p) = \{u \mid wRu\}$. Let $v \in W$ be such that wRv. Then by definition of V, $\mathcal{M}, v \models p$. But, since R doesn't have the property in question, v can't see a world in $V(p)$. Hence, $\mathcal{M}, v \not\models \Diamond p$. And since v was arbitrary, $\mathcal{M}, w \not\models \Diamond(p \to \Diamond p)$. So $\Diamond(p \to \Diamond p)$ is not valid on $\mathcal{F}$.

4. Show that $R^{\mathbf{K4}}$ is transitive: $\forall \Gamma_1, \Gamma_2, \Gamma_3$ if $\Gamma_1 R^{\mathbf{K4}} \Gamma_2$ and $\Gamma_2 R^{\mathbf{K4}} \Gamma_3$, then $\Gamma_1 R^{\mathbf{K4}} \Gamma_3$.

► Solution: Suppose $\Gamma_1 R^{\mathbf{K4}} \Gamma_2$, and $\Gamma_2 R^{\mathbf{K4}} \Gamma_3$. Let $\Box\varphi \in \Gamma_1$. Since $\Box\varphi \to \Box\Box\varphi$ is an instance of the 4 axiom, and since Γ_1 is closed under modus ponens, $\Box\Box\varphi \in \Gamma_1$. Hence, $\Box\varphi \in \Gamma_2$ since $\Gamma_1 R^{\mathbf{K4}} \Gamma_2$. But then $\varphi \in \Gamma_3$, since $\Gamma_2 R^{\mathbf{K4}} \Gamma_3$. So for all $\Box\varphi \in \Gamma_1$, $\varphi \in \Gamma_3$, i.e., $\Gamma_1 R^{\mathbf{K4}} \Gamma_3$.