DEDUCTIONS

PHIL 140A SPRING 2016

---

1. Show that the following propositions are derivable:

(a) $\neg(\varphi \wedge \neg\psi) \to (\varphi \to \psi)$

Proof *(a)*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \qquad [\neg\psi]^2}{\varphi \wedge \neg\psi}\wedge I \qquad [\neg(\varphi \wedge \neg\psi)]^3}{\bot}\to E}{\psi}\,\mathrm{RAA}_2}{\varphi \to \psi}\to I_1}{\neg(\varphi \wedge \neg\psi) \to (\varphi \to \psi)}\to I_3$$

(b) $((\varphi \to \psi) \wedge (\varphi \to \neg\psi)) \to \neg\varphi$

Proof *(b)*:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad \dfrac{[(\varphi \to \psi) \wedge (\varphi \to \neg\psi)]^1}{\varphi \to \psi}\wedge E}{\psi}\to E \qquad \dfrac{[\varphi]^2 \qquad \dfrac{[(\varphi \to \psi) \wedge (\varphi \to \neg\psi)]^1}{\varphi \to \neg\psi}\wedge E}{\neg\psi}\to E}{\bot}\to E}{\neg\varphi}\to I_2}{((\varphi \to \psi) \wedge (\varphi \to \neg\psi)) \to \neg\varphi}\to I_1$$

(c) $\neg(\varphi \to \psi) \to (\psi \to \varphi)$

Proof *(c)*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\varphi]^1 \qquad [\varphi]^2}{\bot}\to E}{\psi}\,\bot}{\varphi \to \psi}\to I_2 \qquad [\neg(\varphi \to \psi)]^3}{\bot}\to E}{\varphi}\,\mathrm{RAA}_1}{\psi \to \varphi}\to I_{\mathrm{vac}}}{\neg(\varphi \to \psi) \to (\psi \to \varphi)}\to I_3$$

(d) $\neg(\varphi \to \psi) \to (\varphi \to \neg\psi)$

Proof *(d)*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg\psi]^1 \qquad [\neg\psi]^2}{\bot}\to E}{\psi}\,\mathrm{RAA}_2}{\varphi\to\psi}\to I_{\mathrm{vac}} \qquad [\neg(\varphi\to\psi)]^3}{\bot}\to E}{\neg\psi}\,\mathrm{RAA}_1}{\dfrac{\varphi\to\neg\psi}{\neg(\varphi\to\psi)\to(\varphi\to\neg\psi)}\to I_3}\to I_{\mathrm{vac}}$$

2. Show that the following rules are derivable in our deduction system:

$$\dfrac{\neg\neg\varphi}{\varphi}\,\neg\neg E \qquad\qquad \dfrac{\varphi\to\psi}{\neg\psi\to\neg\varphi}\,\mathrm{CP}$$

Proof: Suppose you have the following inference in your derivation tree:

$$\dfrac{\begin{matrix}\mathcal{D}\\ \neg\neg\varphi\end{matrix}}{\varphi}\,\neg\neg E$$

Then replace this subtree in your derivation with the following (where n occurs nowhere else in your derivation tree):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \neg\neg\varphi\end{matrix} \qquad [\neg\varphi]^n}{\bot}\to E}{\varphi}\,\mathrm{RAA}_n$$

Likewise if you have the following inference in your derivation tree:

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\to\psi\end{matrix}}{\neg\psi\to\neg\varphi}\,\mathrm{CP}$$

then replace it with:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\to\psi\end{matrix} \qquad [\varphi]^n}{\psi}\to E \qquad [\neg\psi]^{n+1}}{\bot}\to E}{\neg\varphi}\to I_n}{\neg\psi\to\neg\varphi}\to I_{n+1}$$

3. Suppose we replaced RAA with $\neg\neg E$ for only atomic formulas. Show that the full $\neg\neg E$ would still be derivable.

Proof: We need to proceed by induction. We're assuming that the following rules hold for atomic φ (including $\bot$!):

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg E_{\text{at}}$$

So our base case is already taken care of by $\neg\neg E_{\text{at}}$. So suppose $\neg\neg E_\psi$ holds, where ψ is any proper subformula of φ.

First, suppose $\varphi = (\psi \wedge \theta)$. We want to show that we can derive this:

$$\cfrac{\begin{matrix}\mathcal{D}\\ \neg\neg(\psi\wedge\theta)\end{matrix}}{(\psi\wedge\theta)}\ \neg\neg E_\varphi$$

Here's how we do it. The proof doesn't fit on the whole page, so I'll split it into parts. Let $\mathcal{D}_\psi$ be the following derivation tree (where again n occurs nowhere in our derivation tree):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\neg\psi]^n \qquad \cfrac{[(\psi\wedge\theta)]^{n+1}}{\psi}\ \wedge E}{\bot}\ \to E}{\neg(\psi\wedge\theta)}\ \to I_{n+1} \qquad \begin{matrix}\mathcal{D}\\ \neg\neg(\psi\wedge\theta)\end{matrix}}{\bot}\ \to E}{\neg\neg\psi}\ \to I_n}{\psi}\ \neg\neg E_\psi$$

Likewise, let $\mathcal{D}_\theta$ be the following derivation tree:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\neg\theta]^{n+2} \qquad \cfrac{[(\psi\wedge\theta)]^{n+3}}{\theta}\ \wedge E}{\bot}\ \to E}{\neg(\psi\wedge\theta)}\ \to I_{n+3} \qquad \begin{matrix}\mathcal{D}\\ \neg\neg(\psi\wedge\theta)\end{matrix}}{\bot}\ \to E}{\neg\neg\theta}\ \to I_{n+2}}{\theta}\ \neg\neg E_\theta$$

Then we just need to replace

$$\cfrac{\begin{matrix}\mathcal{D}\\ \neg\neg(\psi\wedge\theta)\end{matrix}}{(\psi\wedge\theta)}\ \neg\neg E_\varphi$$

with:

$$\dfrac{\begin{matrix}\mathcal{D}_\psi & \mathcal{D}_\theta \\ \psi & \theta\end{matrix}}{(\psi \wedge \theta)}\wedge I$$

Next, suppose $\varphi = (\psi \to \theta)$. We want to show that we can derive this:

$$\dfrac{\begin{matrix}\mathcal{D} \\ \neg\neg(\psi \to \theta)\end{matrix}}{(\psi \to \theta)}\neg\neg E_\varphi$$

Here's how we do it:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^n \quad [\varphi \to \psi]^{n+1}}{\psi}\to E \quad [\neg\psi]^{n+2}}{\bot}\to E}{\neg(\varphi \to \psi)}\to I_{n+1} \quad \begin{matrix}\mathcal{D} \\ \neg\neg(\varphi \to \psi)\end{matrix}}{\bot}\to E}{\neg\neg\psi}\to I_{n+2}}{\psi}\neg\neg E_\psi}{\varphi \to \psi}\to I_n$$

So by induction, we can rederive $\neg\neg E$ in full. ■