Proofs by Induction

Phil 140A | Spring 2016

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1. Prove that $\sum_{k=1}^{n}(2k-1) = n^2$ for all $n \geqslant 1$.

> **Proof**: First, we show the base case, i.e., the case where $k = 1$. $2k - 1 = (2 \cdot 1) - 1 = 2 - 1 = 1 = 1^2$.
>
> Next, we show the inductive step. Assume as our inductive hypothesis that for some n, $\sum_{k=1}^{n}(2k-1) = n^2$. We want to show that this holds also for $n+1$, i.e., that $\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$. Here's the argument:
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> $$\begin{aligned} \sum_{k=1}^{n+1}(2k-1) &= (2 \cdot (n+1)) - 1 + \sum_{k=1}^{n} 2k - 1 \\ &= (2n+2) - 1 + \sum_{k=1}^{n}(2k-1) \\ &= 2n + 1 + \sum_{k=1}^{n}(2k-1) \\ &= 2n + 1 + n^2 \\ &= (n+1)^2. \end{aligned}$$
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> The second to last step follows by our inductive hypothesis.So by induction, for all n, $\sum_{k=1}^{n}(2k-1) = n^2$. ■

2. Prove that for any set x, if x has exactly n members, then $\mathcal{P}(x)$ has exactly 2^n members (i.e., there are 2^n subsets of x).

> **Proof**: First, we show the base case. The empty set $\varnothing$ has no members. $\mathcal{P}(\varnothing) = \{\varnothing\}$, which has one member. And $2^0 = 1$, so the claim holds.
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> Next, we show the inductive step. Assume as our inductive hypothesis that for some n, if a set x has n elements, then it has 2^n subsets. We want to show that this is also true for $n+1$. Let x be a set with $n+1$ members, say

$a_1, \ldots, a_{n+1}$. We proceed in two parts. First, we'll count the number of subsets of x that do not contain a_{n+1}. Then we'll count the number of subsets of x that do contain a_{n+1}.

First, how many subsets of x are there that don't contain a_{n+1}? This is equivalent to asking how many subsets of $y = \{a_1, \ldots, a_n\}$ there are. But since y only has n members, by our inductive hypothesis, it follows that y has 2^n subsets. So there are 2^n-many subsets of x that do not contain a_{n+1}.

Next, how many subsets of x are there that do contain a_{n+1}. The answer is: *the same as the number of subsets that do not contain a_{n+1}*. For each subset that does contain a_{n+1} corresponds to exactly one subset of x that doesn't contain a_{n+1} (namely, the subset you obtain by removing a_{n+1}). So there are also 2^n subsets of x that do contain a_{n+1}.

Putting these observations together, there are $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of x. So by induction, this is true for all n. ■

3. Define the ***length*** of a propositional formula φ as follows:

$$\begin{aligned} \mathsf{len}(p_i) &= \mathsf{len}(\bot) = 1 \\ \mathsf{len}((\neg\varphi)) &= \mathsf{len}(\varphi) + 1 \\ \mathsf{len}((\varphi \Box \psi)) &= \mathsf{len}(\varphi) + \mathsf{len}(\psi) + 1. \end{aligned}$$

Prove that the number of subformulas in φ is less than or equal to $\mathsf{len}(\varphi)$.

Proof: Let $\#\mathsf{sub}(\varphi)$ stand for the number of subformulas of φ. We proceed by induction on the structure of formulas. First, the base case. Since p_i and $\bot$ only have one subformula (namely themselves), $\#\mathsf{sub}(p_i) = \mathsf{len}(p_i)$ and $\#\mathsf{sub}(\bot) = \mathsf{len}(\bot)$.

Next, we show the inductive steps. There are technically five inductive steps, one for each connective. But since the $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are all of the same form, we'll just present it generically with $\Box$.

First, the $\neg$ case. Assume as our inductive hypothesis that $\#\mathsf{sub}(\varphi) \leqslant \mathsf{len}(\varphi)$. Then $\#\mathsf{sub}((\neg\varphi)) = \#\mathsf{sub}(\varphi) + 1$ (to include $(\neg\varphi)$ itself). And since $\#\mathsf{sub}(\varphi) \leqslant \mathsf{len}(\varphi)$, it follows that $\#\mathsf{sub}((\neg\varphi)) = \#\mathsf{sub}(\varphi) + 1 \leqslant \mathsf{len}(\varphi) + 1 = \mathsf{len}((\neg\varphi))$. Hence, we have shown that $\#\mathsf{sub}((\neg\varphi)) \leqslant \mathsf{len}((\neg\varphi))$.

Next, the $\Box$ case. Assume as our inductive hypothesis that $\#\mathsf{sub}(\varphi) \leqslant \mathsf{len}(\varphi)$ and $\#\mathsf{sub}(\psi) \leqslant \mathsf{len}(\psi)$. Then $\#\mathsf{sub}((\varphi \Box \psi)) \leqslant \#\mathsf{sub}(\varphi) + \#\mathsf{sub}(\psi) + 1$ (remember, it may not be equal if φ and ψ have subformulas in common!). And since $\#\mathsf{sub}(\varphi) \leqslant \mathsf{len}(\varphi)$ and $\#\mathsf{sub}(\psi) \leqslant \mathsf{len}(\psi)$, it follows that $\#\mathsf{sub}((\varphi \Box \psi)) \leqslant \#\mathsf{sub}(\varphi) + \#\mathsf{sub}(\psi) + 1 \leqslant \mathsf{len}(\varphi) + \mathsf{len}(\psi) + 1 = \mathsf{len}((\varphi \Box \psi))$. Hence, we have that $\#\mathsf{sub}((\varphi \Box \psi)) \leqslant \mathsf{len}((\varphi \Box \psi))$. ■

4. For any three formulas φ, ψ, and θ, define the formula $\theta[\psi/\varphi]$ to be the result of replacing every instance of ψ in θ with φ (if there is no occurrence of φ in θ, then $\theta[\psi/\varphi] = \theta$). Show that if φ and ψ are logically equivalent (i.e., φ and ψ have the same truth value on every row of the truth table), then θ and $\theta[\psi/\varphi]$ are logically equivalent. (*Hint*: don't forget the case where $\theta = \varphi$.)

Proof: Either $\theta = \varphi$ or $\theta \neq \varphi$. First, suppose $\theta = \varphi$. Then $\theta[\psi/\varphi] = \varphi[\psi/\varphi] = \psi$. Now, by assumption, φ and ψ are logically equivalent. So $\theta = \varphi$ and $\theta[\psi/\varphi] = \psi$ are logically equivalent.

Now suppose for the rest of the proof that $\theta \neq \varphi$. We must proceed by induction on the structure of θ. For the base cases, suppose first that $\theta = p$ for some atomic p. Since we're assuming that $\theta \neq \varphi$, that means φ can't occur anywhere in p. So $\theta[\psi/\varphi] = \theta$. Hence, θ and $\theta[\psi/\varphi]$ are logically equivalent.

Now for the inductive steps. This time, we can't just schematize the binary connectives as $\Box$. We must separate the inductive steps for each connective. But we'll only show the $\neg$ case for illustration. Throughout, let our inductive hypothesis be the following: for any proper subformula θ' of θ, θ' is logically equivalent to $\theta'[\psi/\varphi]$.

Consider the $\neg$ case. Suppose $\theta = (\neg\,\theta')$. By our inductive hypothesis, that means that θ' and $\theta'[\psi/\varphi]$ are logically equivalent. Now, by the definition of the truth table for $\neg$, for any formula χ, the truth value of $(\neg\,\chi)$ on any row of the truth table is just the opposite of the truth value of χ on that row. Hence, the truth value of $(\neg\,\theta')$ on any row is the opposite of the truth value of θ' on that row. Likewise for $\theta'[\psi/\varphi]$. But if the column of truth values for θ' and $\theta'[\psi/\varphi]$ is the same, and we just flip the truth value on every row in those columns, the resulting columns will still be the same. Hence, $(\neg\,\theta')$ and $(\neg\,\theta'[\psi/\varphi])$ have the same truth value on every row of the truth table.

But we're not done yet! We haven't shown what we want to show. So far, we've shown that $(\neg\,\theta')$ and $(\neg\,\theta'[\psi/\varphi])$ are logically equivalent. But what we *really* wanted to show was that $(\neg\,\theta')$ and $\underline{(\neg\,\theta')[\psi/\varphi]}$ are logically equivalent. The $[\psi/\varphi]$ needs to apply to $(\neg\,\theta')$ not just θ' when θ' is under a $\neg$. This makes a difference. For instance, $p[\bot/(\neg\,p)] = p$, since $(\neg\,p)$ occurs nowhere in p. Hence, $(\neg\,p[\bot/(\neg\,p)]) = (\neg\,p)$. But $(\neg\,p)[\bot/(\neg\,p)] = \bot$. So the scope of $[\psi/\varphi]$ makes a difference.

However, we're assuming that $\theta = (\neg\,\theta')$ is not equal to φ. So we know that $\theta[\psi/\varphi] = (\neg\,\theta')[\psi/\varphi]$ is not equal to ψ. So any replacement of an occurrence of φ with ψ in θ must result from replacing that φ in θ' with ψ and then negating the result. So $\theta[\psi/\varphi] = (\neg\,\theta')[\psi/\varphi] = (\neg\,\theta'[\psi/\varphi])$, which we already showed is equivalent to θ. So now we've shown what we wanted to show: $\neg\,\theta'$ and $(\neg\,\theta')[\psi/\varphi]$ are logically equivalent. ■