SEMANTICS FOR PREDICATE LOGIC

PHIL 140A — SPRING 2016

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1. Carry out the following substitutions:

   (a) $\exists x\ (P(x) \wedge \forall y\ (Q(y) \rightarrow R(x, y)))\ [y/x]$

   (b) $(\exists x\, R(x, c) \vee \forall y\, \neg P(c, x, y))\ [c/x]$

   (c) $(\neg \forall y\, Q(f(x, y)) \wedge f(d, x) = z)\ [f(x, y)/x]$

   (d) $(\neg R(x, f(f(x))) \wedge \exists z\ (f(z) = f(x)))\ [z/f(x)][z/f(f(x))]$

   > **Answer**:
   >
   > (a) $\exists x\ (P(x) \wedge \forall y\ (Q(y) \rightarrow R(x, y)))$
   >
   > (b) $(\exists x\, R(x, c) \vee \forall y\, \neg P(c, c, y))$
   >
   > (c) $(\neg \forall y\, Q(f(x, y)) \wedge f(d, f(x, y)) = z)$
   >
   > (d) $(\neg R(x, f(z)) \wedge \exists z\ (f(z) = f(x)))$

2. Recall from Definition 3.4.4. the following definitions:

$$
\begin{aligned}
Cl(\varphi(z_1, \ldots, z_n)) &\quad := \quad \forall z_1 \cdots \forall z_n\, \varphi(z_1, \ldots, z_n) \\
\mathfrak{A} \models \varphi &\quad \Leftrightarrow \quad \mathfrak{A} \models Cl(\varphi) \\
\models \varphi &\quad \Leftrightarrow \quad \text{for all } \mathfrak{A}, \mathfrak{A} \models \varphi \\
\mathfrak{A} \models \Gamma &\quad \Leftrightarrow \quad \text{for all } \varphi \in \Gamma, \mathfrak{A} \models \varphi \\
\Gamma \models \varphi &\quad \Leftrightarrow \quad \text{for all } \mathfrak{A}, \mathfrak{A} \models \Gamma \Rightarrow \mathfrak{A} \models \varphi.
\end{aligned}
$$

   True or false:

   (a) $\{P(x)\} \models P(y)$.

   > **Answer**: True.
   > **Reason**: This is equivalent to $\{\forall x\, P(x)\} \models \forall y\, P(y)$.

   (b) $\{P(x)\} \models \forall x\, P(x)$.

   > **Answer**: True.
   > **Reason**: This is equivalent to $\{\forall x\, P(x)\} \models \forall x\, P(x)$.

   (c) For all formulas φ, and all models $\mathfrak{A}$, either $\mathfrak{A} \models \varphi$ or $\mathfrak{A} \models \neg \varphi$.

Answer: False.
Example: Let $\mathfrak{A} = (\mathbb{N}, <)$. Consider the formula $x < y$. Then $\mathfrak{A} \not\models x < y$ (since $\mathfrak{A} \not\models \forall x \forall y\ x < y$) and $\mathfrak{A} \not\models \neg(x < y)$ (since $\mathfrak{A} \not\models \forall x \forall y\ \neg\, x < y$).

(d) For all formulas φ and ψ, if $\models \varphi \to \psi$, then $\{\varphi\} \models \psi$.

Answer: True.
Proof: Let $FV(\varphi) - FV(\psi) = \{x_1, \ldots, x_n\}$, $FV(\varphi) \cap FV(\psi) = \{y_1, \ldots, y_m\}$, and $FV(\psi) - FV(\varphi) = \{z_1, \ldots, z_k\}$ (we need to separate the variables that φ and ψ share from the ones they don't share). Suppose for all models $\mathfrak{A}$:

$$\mathfrak{A} \models \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_m \forall z_1 \cdots \forall z_k (\varphi \to \psi).$$

We want to show that for all models $\mathfrak{A}$, if $\mathfrak{A} \models \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_m \varphi$, then $\mathfrak{A} \models \forall y_1 \cdots \forall y_m \forall z_1 \cdots \forall z_k \psi$.

Suppose $\mathfrak{A} \models \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_m \varphi$. Let $a_1, \ldots, a_n \in |\mathfrak{A}|$ be such that $\mathfrak{A} \models \forall y_1 \cdots \forall y_m \varphi[\bar{a}_1/x_1] \cdots [\bar{a}_n/x_n]$. By our supposition, we can universally instantiate to get:

$$\mathfrak{A} \models \forall y_1 \cdots \forall y_m \forall z_1 \cdots \forall z_k (\varphi \to \psi)[\bar{a}_1/x_1] \cdots [\bar{a}_n/x_n].$$

But since none of $x_1, \ldots, x_n$ occur in ψ, this is equivalent to:

$$\mathfrak{A} \models \forall y_1 \cdots \forall y_m \forall z_1 \cdots \forall z_k (\varphi[\bar{a}_1/x_1] \cdots [\bar{a}_n/x_n] \to \psi).$$

Now, suppose $b_1, \ldots, b_m, c_1, \ldots, c_k \in |\mathfrak{A}|$. Then by universal instantiation:

$$\mathfrak{A} \models \varphi[\bar{a}_1/x_1] \cdots [\bar{a}_n/x_n][\bar{b}_1/y_1] \cdots [\bar{b}_m/y_m]$$
$$\mathfrak{A} \models (\varphi[\bar{a}_1/x_1] \cdots [\bar{a}_n/x_n] \to \psi)[\bar{b}_1/y_1] \cdots [\bar{b}_m/y_m][\bar{c}_1/z_1] \cdots [\bar{c}_k/z_k].$$

But since none of $z_1, \ldots, z_k$ occur in $\{y_1, \ldots, y_m\}$, this second line is equivalent to:

$$\mathfrak{A} \models (\varphi[\bar{a}_1/x_1] \cdots [\bar{a}_n/x_n][\bar{b}_1/y_1] \cdots [\bar{b}_m/y_m] \to \psi[\bar{b}_1/y_1] \cdots [\bar{b}_m/y_m][\bar{c}_1/z_1] \cdots [\bar{c}_k/z_k]).$$

Hence:

$$\mathfrak{A} \models \psi[\bar{b}_1/y_1] \cdots [\bar{b}_m/y_m][\bar{c}_1/z_1] \cdots [\bar{c}_k/z_k].$$

Since $b_1, \ldots, b_m, c_1, \ldots, c_k$ were arbitrary, $\mathfrak{A} \models \forall y_1 \cdots \forall y_m \forall z_1 \cdots \forall z_k \psi$. ■

(e) For all formulas φ and ψ, if $\{\varphi\} \models \psi$, then $\models \varphi \to \psi$.

> **Answer**: False.
>
> **Example**: Let $\varphi = P(x)$ and $\psi = P(y)$. Then it's true that for all models $\mathfrak{A}$, if $\mathfrak{A} \models P(x)$ (i.e., $\mathfrak{A} \models \forall x\, P(x)$), then $\mathfrak{A} \models P(y)$ (i.e., $\mathfrak{A} \models \forall y\, P(y)$). However, it's not true that for all models $\mathfrak{A}$, $\mathfrak{A} \models P(x) \to P(y)$. This is equivalent to $\mathfrak{A} \models \forall x \forall y\ (P(x) \to P(y))$, which is equivalent to $\mathfrak{A} \models \exists x\, P(x) \to \forall y\, P(y)$. But not every model makes this true.

(f) For all formulas φ, $\models \varphi$ iff $\varnothing \models \varphi$.

> **Answer**: True.
>
> **Reason**: $\models \varphi$ iff for all $\mathfrak{A}$, $\mathfrak{A} \models \varphi$. But this holds iff for all $\mathfrak{A}$, if $\mathfrak{A} \models \psi$ for every $\psi \in \varnothing$, then $\mathfrak{A} \models \varphi$. And so this is holds iff $\varnothing \models \varphi$.

3. Which of the following is true for all models $\mathfrak{A}$ and all formulas φ and ψ?

(a) $\mathfrak{A} \models \neg\varphi$ iff $\mathfrak{A} \not\models \varphi$. **False.**

(b) $\mathfrak{A} \models \varphi \wedge \psi$ iff $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \psi$. **True.**

(c) $\mathfrak{A} \models \varphi \vee \psi$ iff $\mathfrak{A} \models \varphi$ or $\mathfrak{A} \models \psi$. **False.**

(d) $\mathfrak{A} \models \varphi \to \psi$ iff $\mathfrak{A} \models \varphi$ only if $\mathfrak{A} \models \psi$. **False.**

(e) $\mathfrak{A} \models \varphi \leftrightarrow \psi$ iff [$\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \psi$]. **False.**

(f) $\mathfrak{A} \models \forall x\, \varphi$ iff for all $a \in |\mathfrak{A}|$, $\mathfrak{A} \models \varphi[\bar{a}/x]$. **True.**

(g) $\mathfrak{A} \models \exists x\, \varphi$ iff for some $a \in |\mathfrak{A}|$, $\mathfrak{A} \models \varphi[\bar{a}/x]$. **False.**