SET THEORY

PHIL 140A SPRING 2016

DEFINITIONS

Name	Notation	Definition		
Axiom of Extensionality		$x = y$	$\leftrightarrow$	$\forall a\ (a \in x \leftrightarrow a \in y)$
Restricted Quantifiers	$\forall a \in x$	$\forall a \in x\ \varphi$	$=$	$\forall a\ (a \in x \to \varphi)$
	$\exists a \in x$	$\exists a \in x\ \varphi$	$=$	$\exists a\ (a \in x \land \varphi)$
Subset	$\subseteq$	$x \subseteq y$	$\leftrightarrow$	$\forall a\ (a \in x \to a \in y)$
Proper Subset	$\subset$	$x \subset y$	$\leftrightarrow$	$x \subseteq y \land x \neq y$
Intersection	$\cap$	$a \in x \cap y$	$\leftrightarrow$	$(a \in x \land a \in y)$
		$x \cap y$	$=$	$\{a \mid a \in x \land a \in y\}$
Union	$\cup$	$a \in x \cup y$	$\leftrightarrow$	$(a \in x \lor a \in y)$
		$x \cup y$	$=$	$\{a \mid a \in x \lor a \in y\}$
Complement	$-$	$a \in x - y$	$\leftrightarrow$	$(a \in x \land a \notin y)$
		$x - y$	$=$	$\{a \mid a \in x \land a \notin y\}$
Power set	$\mathcal{P}$	$a \in \mathcal{P}(x)$	$\leftrightarrow$	$a \subseteq x$
		$\mathcal{P}(x)$	$=$	$\{a \mid a \subseteq x\}$
Ordered Pair	(a, b)	(a, b)	$=$	$\{\{a\}, \{a, b\}\}$
Ordered n-Tuple		$(a_1, \ldots, a_n)$	$=$	$((a_1, \ldots, a_{n-1}), a_n)$
Big Union	$\bigcup$	$a \in \bigcup x$	$\leftrightarrow$	$\exists y\ (a \in y \land y \in x)$
		$\bigcup x$	$=$	$\{a \mid \exists y\ (a \in y \land y \in x)\}$
		$\bigcup_{i \in I} x_i$	$=$	$\bigcup \{x_i \mid i \in I\}$
Big Intersection	$\bigcap$	$a \in \bigcap x$	$\leftrightarrow$	$\forall y\ (y \in x \to a \in y)$
		$\bigcap x$	$=$	$\{a \mid \forall y\ (y \in x \to a \in y)\}$
		$\bigcap_{i \in I} x_i$	$=$	$\bigcap \{x_i \mid i \in I\}$
Cartesian Product	$\times$	$x \times y$	$=$	$\{(a, b) \mid a \in x \land b \in y\}$
Domain	dom	$\mathrm{dom}(R)$	$=$	$\{a \mid \exists b\ ((a, b) \in R)\}$
Range	ran	$\mathrm{ran}(R)$	$=$	$\{b \mid \exists a\ ((a, b) \in R)\}$
Composition	$\circ$	$(g \circ f)(a)$	$=$	$g(f(a))$
Equivalence Class	$[a]_R$	$[a]_R$	$=$	$\{b \mid R(a, b)\}$

Properties of Relations

Name	Definition
Reflexive	$\forall a\, R(a,a)$
Irreflexive	$\forall a\, \neg R(a,a)$
Symmetric	$\forall a \forall b\ (R(a,b) \to R(b,a))$
Asymmetric	$\forall a \forall b\ (R(a,b) \to \neg R(b,a))$
Anti-symmetric	$\forall a \forall b\ ((R(a,b) \wedge R(b,a)) \to a = b)$
Transitive	$\forall a \forall b \forall c\ ((R(a,b) \wedge R(b,c)) \to R(a,c))$
Euclidean	$\forall a \forall b \forall c\ ((R(a,b) \wedge R(a,c)) \to R(b,c))$
Connected	$\forall a \forall b\ (a \neq b \to (R(a,b) \vee R(b,a)))$

- A relation is an ***equivalence relation*** iff it's reflexive, symmetric, and transitive.
- (x, R) is a ***partial order*** iff $R \subseteq x \times x$ and is reflexive, anti-symmetric, and transitive.
- A partial order $(x, \leqslant)$ has a is a ***minimal element*** in y if there's no $b \in y$ where $b < a$.
- $(x, \leqslant)$ is ***well-founded*** if every nonempty $y \subseteq x$ has a minimal element.

Examples

Note: these proofs are purposely wordy so that my reasoning is clear. In the first couple of problem sets, it's good idea to go step-by-step and explain your reasoning clearly than to skip a bunch of steps. However, for the problem set, you don't necessarily need this much wordiness; as long as your reasoning is clearly stated, that's okay.

Exercise. Prove that $x \cup y = y \cup x$.

Proof: By the Axiom of Extensionality, it suffices to show that:

$$\forall a\ (a \in x \cup y \leftrightarrow a \in y \cup x)\,.$$

Suppose first that $a \in x \cup y$ for some arbitrary a. By the definition of union, that means $a \in x \vee a \in y$. But by propositional logic, this is equivalent to $a \in y \vee a \in x$. So by the definition of union again, that means $a \in y \cup x$. Hence, if $a \in x \cup y$, then $a \in y \cup x$, i.e., $a \in x \cup y \to a \in y \cup x$.

The converse is a symmetric argument. Suppose that $a \in y \cup x$. By the definition of union, that means $a \in y \vee a \in x$. But by propositional logic, this is equivalent to $a \in x \vee a \in y$. So by the definition of union again, that means $a \in x \cup y$. Hence, if $a \in y \cup x$, then $a \in x \cup y$, i.e., $a \in y \cup x \to a \in x \cup y$.

Putting these two together by propositional logic, we have $a \in x \cup y \leftrightarrow a \in y \cup x$. And since a was arbitrary, it follows that $\forall a\, (a \in x \cup y \leftrightarrow a \in y \cup x)$, which we said by the Axiom of Extensionality implies that $x \cup y = y \cup x$. ■

Exercise. Prove $x \cup y = y \cup x$ *without* the Axiom of Extensionality.

Hint: use the alternative definition of union from the reading:

$$z = x \cup y \leftrightarrow \forall a \; (a \in z \leftrightarrow (a \in x \vee a \in y)) \, .$$

Exercise. A relation is an ***equivalence relation*** if it's reflexive, symmetric, and transitive. Prove that a relation R is an equivalence relation iff it's reflexive and euclidean

Proof: We need to show two things, namely the left-to-right direction and the right-to-left direction:

($\Rightarrow$) If R is an equivalence relation, then it's reflexive and euclidean.

($\Leftarrow$) If R is reflexive and euclidean, then it's an equivalence relation.

It's easiest to show each direction separately.

($\Rightarrow$) Suppose R is an equivalence relation. By definition, that means R is reflexive, symmetric, and transitive. So we just need to show that it's euclidean, i.e., that $\forall a \, \forall b \, \forall c \; ((R(a, b) \wedge R(a, c)) \rightarrow R(b, c))$. Let a, b, and c be arbitrary elements such that $R(a, b)$ and $R(a, c)$. We want to show that $R(b, c)$. By the symmetry of R, it follows from $R(a, b)$ that $R(b, a)$. But since $R(b, a)$ and $R(a, c)$, it follows from the transitivity of R that $R(b, c)$, which is what we want. So for any arbitrary elements a, b, and c, if $R(a, b)$ and $R(a, c)$, then $R(b, c)$, i.e., R is euclidean. ✓

($\Leftarrow$) Suppose R is reflexive and euclidean. We want to show that R is an equivalence relation, i.e., that R is reflexive, symmetric, and transitive. R is reflexive by hypothesis, so we just need to show symmetry and transitivity.

First, symmetry. Let a and b be arbitrary elements such that $R(a, b)$. We want to show that $R(b, a)$. By the reflexivity of R, $R(a, a)$. But since $R(a, b)$ and $R(a, a)$, it follows by the euclidean-ness of R that $R(b, a)$, which is what we want. So for any arbitrary elements a and b, if $R(a, b)$, then $R(b, a)$, i.e., R is symmetric.

Second, transitivity. Let a, b, and c be arbitrary elements such that $R(a, b)$ and $R(b, c)$. We want to show that $R(a, c)$. By the symmetry of R and since $R(a, b)$, it follows that $R(b, a)$. But then since $R(b, a)$ and $R(b, c)$, by the euclidean-ness of R, it follows that $R(a, c)$, which is what we want. So for any arbitrary a, b, and c, if $R(a, b)$ and $R(b, c)$, then $R(a, c)$, i.e., R is transitive. ✓ ∎