Truth Tables

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1 Basic Truth Tables

α	β	$\alpha \,\&\, \beta$
T	T	T
T	F	F
F	T	F
F	F	F

α	β	$\alpha \vee \beta$
T	T	T
T	F	T
F	T	T
F	F	F

α	β	$\alpha \supset \beta$
T	T	T
T	F	F
F	T	T
F	F	T

α	β	$\alpha \leftrightarrow \beta$
T	T	T
T	F	F
F	T	F
F	F	T

α	$\sim \alpha$
T	F
F	T

Note: These are best thought of as *rules* for evaluating more complicated truth tables.

The rule tells you how to fill out the rows of the truth table for a statement of PL given that the Greek letters are replaced with that statements' immediate constituents.

For instance, if your statement is a logical disjunction, the disjunction table above says "In rows where one of α or β is true, their disjunction is true. But in rows where both α and β are false, their disjunction is false."

If you have hesitations about the conditional table, you're not alone. We'll discuss conditionals more carefully soon, but for now, just to get a feel for how truth tables involving '$\supset$' work, you should try to set this worry aside.

Also, if you have Prof. Yalcin's worry regarding the use/mention distinction when we say things like '$\alpha \,\&\, \beta$', all I can say is be patient. For now, you'll have to forgive my mistakes, as we don't have the right tools yet appropriately deal with this. Context should make my meaning clear.

2 Complicated Truth Tables

The easiest way to approach more complicated truth tables is to break up the problem into simpler parts. For instance, suppose we want to construct a truth table for '$P \vee (Q \,\&\, \sim P)$'. Our first step is (always!) to include columns that account for the simple statement letters occurring in our complex statement.

P	Q
T	T
T	F
F	T
F	F

We then look for the next smallest component in '$P \vee (Q \,\&\, \sim P)$', which is '$\sim P$', and we include a column for it next.

P	Q	$\sim P$
T	T	F
T	F	F
F	T	T
F	F	T

We used the negation table in Section 1 where 'α' is replaced with 'P'. Next, we add a column for '$Q \,\&\, \sim P$'.

P	Q	$\sim P$	$Q \,\&\, \sim P$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Now, we use the conjunction table, replacing 'α' with 'Q' and 'β' with '$\sim P$'. Remember the intuitive definition: '$Q \,\&\, \sim P$' is true if both 'Q' and '$\sim P$' are true; otherwise, '$Q \,\&\, \sim P$' is false.

We then complete the truth table, by considering '$P \vee (Q \,\&\, \sim P)$'. We use the disjunction table where we replace 'α' with 'P' and 'β' with '$Q \,\&\, \sim P$'.

P	Q	$\sim P$	$Q \,\&\, \sim P$	$P \vee (Q \,\&\, \sim P)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

Notice something odd though: '$P \vee (Q \,\&\, \sim P)$' has *exactly the same* truth table as '$P \vee Q$'. What this shows is that '$P \vee Q$' and '$P \vee (Q \,\&\, \sim P)$' are **logically equivalent**. Roughly speaking, this means that, for the sake of argument, it doesn't matter which sentence you use.

P	Q	$P \vee Q$	$P \vee (Q \,\&\, \sim P)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Suppose now we want the truth table for ‘ $R \supset (P \vee (Q \;\&\; \sim P))$ ’. How do we expand our truth table? Well, for one thing, we need to add more rows.

FACT
If you want to construct a truth table for the statement α of PL, and α has n simple statement letters, you'll need 2^n rows. For instance, for the statement ‘ $P \;\&\; (Q \;\&\; (R \;\&\; S))$ ’, you'll need $2^4 = 16$ rows.

So we need to add more rows to our truth table in order to accomodate the presence of ‘ R ’. Using the formula above, we'll need $2^3 = 8$ rows.

P	Q	R	$P \vee (Q \;\&\; \sim P)$	$R \supset (P \vee (Q \;\&\; \sim P))$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

To ensure you hit each combination, you can follow the following algorithm. For the first simple statement letter, make the first half of the rows T-rows, and the second half F-rows. For the second statement letter, make the first *quarter* T-rows, the second quarter F-rows, the third quarter T-rows, and the fourth quarter F-rows. For the third simple sentence letter, make the first *eighth* T-rows, the second eighth F-rows,. . . and so on.

So how do we fill out that ‘ $P \vee (Q \;\&\; \sim P)$ ’ column? One method is to use the last truth table on the previous page. Another method is to notice, as we did above, that ‘ $P \vee (Q \;\&\; \sim P)$ ’ has the same truth table as ‘ $P \vee Q$ ’. So we can just use the truth table for ‘ $P \vee Q$ ’ to fill out that column here. Either way, this table tells us that ‘ $P \vee (Q \;\&\; \sim P)$ ’ is false when both ‘ P ’ and ‘ Q ’ are false; otherwise it's true. Using this rule, we can fill out the column accordingly.:

P	Q	R	$P \vee (Q \;\&\; \sim P)$	$R \supset (P \vee (Q \;\&\; \sim P))$
T	T	T	T	
T	T	F	T	
T	F	T	T	
T	F	F	T	
F	T	T	T	
F	T	F	T	
F	F	T	F	
F	F	F	F	

To fill out the last column, we recall the conditional page at the beginning: $\alpha \supset \beta$ is false if α is true and β is false; otherwise it's true. So in this example, ' $R \supset (P \vee (Q \, \& \, \sim P))$ ' is false if ' R ' is true and ' $P \vee (Q \, \& \, \sim P)$ ' is false. This only happens in one row:

P	Q	R	$P \vee (Q \, \& \, \sim P)$	$R \supset (P \vee (Q \, \& \, \sim P))$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Notice that because ' $P \vee (Q \, \& \, \sim P)$ ' is logically equivalent to ' $P \vee Q$ ', it follows that ' $R \supset (P \vee (Q \, \& \, \sim P))$ ' is logically equivalent to ' $R \supset (P \vee Q)$ '. In other words, you can *substitute* logically equivalent statements into larger statements without affecting truth tables or logical validity. So if I asked you to now write a truth table for ' $R \supset (P \vee Q)$ ', the hard work is already done: because the columns for ' $P \vee (Q \, \& \, \sim P)$ ' and ' $P \vee Q$ ' will be the same, you just copy the truth table for ' $R \supset (P \vee (Q \, \& \, \sim P))$ ' under the column ' $R \supset (P \vee Q)$ '.

How about making a truth table for ' $Q \leftrightarrow (\sim P \, \& \, (R \supset \sim Q))$ '? As an exercise, write the table below yourself.

3 Validity and Truth Tables

Okay, so what good are truth tables? Well, for one thing, they can help us determine whether or not an argument formalized in PL is valid.

THEOREM
Suppose an argument has premises $\alpha_1, \ldots, \alpha_n$ and conclusion β in PL. Then the argument

$$\alpha_1, \ldots, \alpha_n \therefore \beta$$

is formally valid **if and only if** there is no row in a truth table containing $\alpha_1, \ldots, \alpha_n$ and β in which all of $\alpha_1, \ldots, \alpha_n$ are true but β is false.

Consider, for instance, the argument:

$$P, P \supset \sim Q \therefore \sim Q$$

It's pretty easy to recognize this argument as formally valid (it's just *modus ponens*). But we can also come to that conclusion just by truth tables.

To determine whether this argument is formally valid, we can create a truth table containing the premises and the conclusion, and search for a row where 'P' and '$P \supset \sim Q$' are true and yet '$\sim Q$' is false. If there is such a row, the argument is not formally valid; otherwise, it is formally valid.

P	Q	$\sim Q$	$P \supset \sim Q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

So, first question, in which rows are both 'P' and '$P \supset \sim Q$' true? Conveniently, there's only one such row, namely the second.

P	Q	$\sim Q$	$P \supset \sim Q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Is this a row in which '$\sim Q$' is false? **No.** Therefore, there is no row in the truth table in which the premises are all true but the conclusion is false. So by our theorem above, the argument is formally valid.

Let's do another example. What about the following argument?

$$\sim P \therefore Q \supset R$$

This seems like it should be obviously invalid. Let's prove it with truth tables.

P	Q	R	$\sim P$	$Q \supset R$
T	T	T	F	T
T	T	F	F	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

Again, if there's a row in which all of the premises are true, but the conclusion is false, then the argument is not formally valid. There are four rows in which our premise is true, but in one of those rows (highlighted), the conclusion is false. So the argument is invalid (but just barely. . . strange).

How about a more complicated example? Consider the argument:

$$P \supset\sim (Q \supset R), \sim Q \therefore \sim P.$$

P	Q	R	$Q \supset R$	$\sim (Q \supset R)$	$P \supset\sim (Q \supset R)$	$\sim Q$	$\sim P$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Formally valid or not?