Equivalences in PL

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Basic Tautologies:
$\alpha \lor \sim\alpha$
$\alpha \supset \alpha$
$\sim$(any contradiction)

Basic Contradictions:
$\alpha \;\&\; \sim\alpha$
$\alpha \leftrightarrow \sim\alpha$
$\sim$(any tautology)

The following are all tautologies.

De Morgan's Laws:
$\sim(\alpha \lor \beta) \leftrightarrow (\sim\alpha \;\&\; \sim\beta)$
$\sim(\alpha \;\&\; \beta) \leftrightarrow (\sim\alpha \lor \sim\beta)$

Double Negation:
$\sim\sim\alpha \leftrightarrow \alpha$

Distribution:
$(\alpha \;\&\; (\beta \lor \gamma)) \leftrightarrow ((\alpha \;\&\; \beta) \lor (\alpha \;\&\; \gamma))$
$(\alpha \lor (\beta \;\&\; \gamma)) \leftrightarrow ((\alpha \lor \beta) \;\&\; (\alpha \lor \gamma))$

Definition of Conditional:
$(\alpha \supset \beta) \leftrightarrow (\sim\alpha \lor \beta)$

Associativity:
$(\alpha \;\&\; (\beta \;\&\; \gamma)) \leftrightarrow ((\alpha \;\&\; \beta) \;\&\; \gamma)$
$(\alpha \lor (\beta \lor \gamma)) \leftrightarrow ((\alpha \lor \beta) \lor \gamma)$
$(\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) \leftrightarrow ((\alpha \leftrightarrow \beta) \leftrightarrow \gamma)$

Commutativity:
$(\alpha \;\&\; \beta) \leftrightarrow (\beta \;\&\; \alpha)$
$(\alpha \lor \beta) \leftrightarrow (\beta \lor \alpha)$
$(\alpha \leftrightarrow \beta) \leftrightarrow (\beta \leftrightarrow \alpha)$

Idempotence:
$(\alpha \;\&\; \alpha) \leftrightarrow \alpha$
$(\alpha \lor \alpha) \leftrightarrow \alpha$

Note that none of the equivalences on the right apply to '$\supset$'.

Import/Export:
$(\alpha \supset (\beta \supset \gamma)) \leftrightarrow ((\alpha \;\&\; \beta) \supset \gamma)$
$(\alpha \supset (\beta \supset \gamma)) \leftrightarrow (\beta \supset (\alpha \supset \gamma))$
$(\alpha \supset (\beta \leftrightarrow \gamma)) \leftrightarrow ((\alpha \;\&\; \beta) \leftrightarrow (\alpha \;\&\; \gamma))$
$((\alpha \lor \beta) \supset \gamma) \leftrightarrow ((\alpha \supset \gamma) \;\&\; (\beta \supset \gamma))$

Transitivity:
$((\alpha \supset \beta) \;\&\; (\beta \supset \gamma)) \supset (\alpha \supset \gamma)$
$((\alpha \leftrightarrow \beta) \;\&\; (\beta \leftrightarrow \gamma)) \supset (\alpha \leftrightarrow \gamma)$

Disjunctive Syllogism:
$((\alpha \lor \beta) \;\&\; \sim\alpha) \supset \beta$

Strengthening of Antecedent:
$(\alpha \supset \gamma) \supset ((\alpha \;\&\; \beta) \supset \gamma)$

Weakening of Consequent:
$(\alpha \supset \gamma) \supset (\alpha \supset (\beta \lor \gamma))$

Contraposition:
$(\alpha \supset \beta) \leftrightarrow (\sim\beta \supset \sim\alpha)$
$(\alpha \leftrightarrow \beta) \leftrightarrow (\sim\alpha \leftrightarrow \sim\beta)$
$\sim(\alpha \leftrightarrow \beta) \leftrightarrow (\alpha \leftrightarrow \sim\beta)$

Paradoxes of Material Conditional:

$\alpha \supset (\beta \vee \sim\beta)$	$(\alpha \supset \beta) \vee (\alpha \supset \sim\beta)$
$(\alpha \,\&\, \sim\alpha) \supset \beta$	$(\alpha \supset \beta) \vee (\sim\alpha \supset \beta)$
$(\alpha \supset \beta) \vee (\beta \supset \alpha)$	$\alpha \supset (\beta \supset \alpha)$

Note: What these "paradoxes" show is that you *cannot* read '$\alpha \supset \beta$' as 'α *implies* β', or as 'α *entails* β'. The material conditional is a *very* weak conditional: it just says that either the antecedent *happens* to be false or the consequent *happens* to be true. There needn't be any connection between the two.

Furthermore, we cannot read '$\supset$' as a counterfactual (or subjunctive) conditional. Consider the following pairs of sentences:

(1) If Oswald didn't shoot JFK, then someone else did.

(2) If Oswald hadn't shot JFK, then someone else would have.

The former is an *indicative* conditional, whereas the latter is a *counterfactual* conditional. The first seems true: we know *someone* shot JFK, so if it wasn't Oswald, it had to be someone else. But the second (assuming the official investigations are accurate) is false: Oswald acted alone, so if he hadn't shot JFK, it seems no one else would have.

There are a number of reasons why '$\supset$' can never be a counterfactual conditional. Here's one, as noted in David Lewis's *Counterfactuals* (p. 10). If '$\alpha \supset \gamma$' is true, then for any β, so is '$(\alpha \,\&\, \beta) \supset \gamma$'. That is, we're allowed to strengthen the antecedent without losing truth. By contrast, we can't do this with counterfactuals. While 'If Oswald hadn't shot JFK, then someone else would have shot JFK' seems fine, 'If Oswald hadn't shot JFK but Oswald had a partner, then someone else would have shot JFK' doesn't. But then 'If Oswald hadn't shot JFK but Oswald had a partner who died the day before, then someone else would have shot JFK' seems fine again. And so on...

Still, despite this, '$\supset$' can be a very useful connective to consider in logic. For instance, *modus ponens*, *modus tollens*, and conditional proof both seem valid for '$\supset$'. You can check this via its truth table:

α	β	$\alpha \supset \beta$
T	T	T
T	F	F
F	T	T
F	F	T

To help remember this table, suppose you're at a bar, and you see four people drinking. The first is drinking alcohol and is 21. The second is drinking alcohol but is 18. The third is drinking coke, and is 21. The fourth is drinking coke and is 18. If the law is "You may drink alcohol only if you're 21", who is breaking the law?