Equivalences in RPL

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Notation: if φ^v is an open sentence, then φ^u is the open sentence that results from replacing every instance of v in φ^v with u.

Arbitrariness of Variables:

If u doesn't occur anywhere in the open sentence φ^v, then:

$(\exists v)\varphi^v \dashv\vDash (\exists u)\varphi^u$

$(\forall v)\varphi^v \dashv\vDash (\forall u)\varphi^u$,

Quantifier Exchange:

$\sim(\exists v)\varphi^v \dashv\vDash (\forall v)\sim\varphi^v$

$\sim(\forall v)\varphi^v \dashv\vDash (\exists v)\sim\varphi^v$

$\sim(\exists v)\sim\varphi^v \dashv\vDash (\forall v)\varphi^v$

$\sim(\forall v)\sim\varphi^v \dashv\vDash (\exists v)\varphi^v$

Null Quantification:

If v doesn't occur anywhere in α, then:

$\alpha \,\&\, (\forall v)\varphi^v \dashv\vDash (\forall v)(\alpha \,\&\, \varphi^v)$

$\alpha \,\&\, (\exists v)\varphi^v \dashv\vDash (\exists v)(\alpha \,\&\, \varphi^v)$

$\alpha \vee (\forall v)\varphi^v \dashv\vDash (\forall v)(\alpha \vee \varphi^v)$

$\alpha \vee (\exists v)\varphi^v \dashv\vDash (\exists v)(\alpha \vee \varphi^v)$

$\alpha \supset (\forall v)\varphi^v \dashv\vDash (\forall v)(\alpha \supset \varphi^v)$

$\alpha \supset (\exists v)\varphi^v \dashv\vDash (\exists v)(\alpha \supset \varphi^v)$

$(\forall v)\varphi^v \supset \alpha \dashv\vDash (\exists v)(\varphi^v \supset \alpha)$

$(\exists v)\varphi^v \supset \alpha \dashv\vDash (\forall v)(\varphi^v \supset \alpha)$

Conjunction:

$(\exists v)(\varphi^v \,\&\, \psi^v) \not\dashv\vDash (\exists v)\varphi^v \,\&\, (\exists v)\psi^v$

$(\forall v)(\varphi^v \,\&\, \psi^v) \dashv\vDash (\forall v)\varphi^v \,\&\, (\forall v)\psi^v$

Disjunction:

$(\exists v)(\varphi^v \vee \psi^v) \dashv\vDash (\exists v)\varphi^v \vee (\exists v)\psi^v$

$(\forall v)(\varphi^v \vee \psi^v) \dashv\nvDash (\forall v)\varphi^v \vee (\forall v)\psi^v$

Conditional:

$(\forall v)(\varphi^v \supset \psi^v) \not\dashv\vDash (\forall v)\varphi^v \supset (\forall v)\psi^v$

$(\exists v)(\varphi^v \supset \psi^v) \dashv\vDash (\forall v)\varphi^v \supset (\exists v)\psi^v$

Any of the rules above can be applied to the parts as well as the wholes. For instance, '$(\forall x)(F^1x \supset \sim(\exists y)R^2xy)$' is logically equivalent to '$(\forall x)(F^1x \supset (\forall y)\sim R^2xy)$' by **Quantifier Exchange**. Similarly, by **Null Quantification**, '$(\forall x)(F^1x \supset (R^2xx \,\&\, (\exists y)R^2xy))$' is logically equivalent to '$(\forall x)(F^1x \supset (\exists y)(R^2xx \,\&\, R^2xy))$'. (Note the α-part doesn't have to be a closed sentence.)

Pay attention to the variable restrictions! For instance, you can't use **Variable Replacement** to say that '$(\forall x)(F^1x \supset (\exists y)R^2xy)$' is logically equivalent to '$(\forall x)(F^1x \supset (\exists x)R^2xx)$', according to the variable rule. The latter is not even a well-formed formula!