Truth-Functional Equivalences

In what follows, $A, B, C, \ldots$ can be simple *or* complex sentences. Here, "$A \Leftrightarrow B$" should be read as "A is equivalent to B," i.e. A has the same truth table as B. The symbol '$\Leftrightarrow$' is *not* a connective!

Tautologies (always $\top$):
$A \vee -A$
$A \supset A$
$A \supset (B \supset A)$
$-$(any contradiction)

Contradictions (always $\perp$):
$A \,.\, -A$
$A \equiv -A$
$A \,.\, (A \supset B) \,.\, -B$
$-$(any tautology)

De Morgan's Laws:
$-(A \vee B) \Leftrightarrow (-A \,.\, -B)$
$-(A \,.\, B) \Leftrightarrow (-A \vee -B)$

Double Negation:
$- - A \Leftrightarrow A$

Distribution:
$A \,.\, (B \vee C) \Leftrightarrow (A \,.\, B) \vee (A \,.\, C)$
$A \vee (B \,.\, C) \Leftrightarrow (A \vee B) \,.\, (A \vee C)$

Associativity:
$A \,.\, (B \,.\, C) \Leftrightarrow (A \,.\, B) \,.\, C$
$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$
$A \equiv (B \equiv C) \Leftrightarrow (A \equiv B) \equiv C$

Commutativity:
$A \,.\, C \Leftrightarrow B \,.\, A$
$A \vee C \Leftrightarrow B \vee A$

Idempotence:
$A \,.\, A \Leftrightarrow A$
$A \vee A \Leftrightarrow A$

Manipulating Conditionals:
$A \supset (B \supset C) \Leftrightarrow B \supset (A \supset C)$
$A \supset (B \supset C) \Leftrightarrow (A \,.\, B) \supset C$
$A \supset B \Leftrightarrow -B \supset -A$
$A \equiv B \Leftrightarrow -A \equiv -B$

Paradoxes of Material Conditional:
$A \supset (B \vee -B)$
$(A \,.\, -A) \supset B$
$(A \supset B) \vee (B \supset A)$
$A \supset (B \supset A)$

Note: You *cannot* read "$A \supset B$" as "A *implies* B" or "A *entails* B," except in very special circumstances (e.g. when doing mathematics, maybe).

Translation Guide for Conditionals:

$\underline{A \supset B}$	$\underline{B \supset A}$	$\underline{A \equiv B}$
A only if B	A if B	A if and only if B
A only when B	A when B	A iff B
If A, then B	A given B	A when and only when B
B unless $-A$	A provided (that) B	
A just in case B [1]		

Definition (Disjunctive Normal Form)

A sentence is in **disjunctive normal form** iff it is the disjunction of conjunctions of atomic and negated atomic sentences.

Example:
"$(-p\ .\ -q\ .\ r) \vee (-p\ .\ q\ .\ -r) \vee (p\ .\ q\ .\ r)$" is in disjunctive normal form.
"$((-p \vee -q \vee r)\ .\ (-p \vee q\ .\ -r)) \vee -(p\ .\ q\ .\ r)$" is not.

Theorem

Any (truth-functional) connective you can define can be redefined using '$-$', '$.$', and '$\vee$' in disjunctive normal form.

▶ **Proof (Idea):** As in section: Conjoin the atomics and negated atomics for each $\top$-row, and then disjoin each of these conjuncts. Example, define a tertiary connective ∇ as follows:

p	q	r	$\nabla(p,q,r)$
$\top$	$\top$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$

In this case, "$\nabla(p,q,r)$" is equivalent to "$(p\ .\ -q\ .\ -r) \vee (-p\ .\ q\ .\ -r) \vee (-p\ .\ -q\ .\ -r)$".

□

[1] This is a weird case. Ordinary English often translates "just in case" or "just in the case where" as "$\supset$", but also sometimes as "$\equiv$". In fact, mathematicians and philosophers often translate them as "$\equiv$". Similarly, mathematicians also say "if" instead of "if and only if" when defining new terminology.

COROLLARY

Any (truth-functional) sentence can be written in disjunctive normal form.

▶ **PROOF (IDEA):** If A is a sentence, write out its truth table. Then construct a sentence in disjunctive normal form with the same truth table, as above. □