Properties of Implication

Legend			
Symbols	**English**	**Symbols**	**English**
$A \models B$	A implies B	$A \models$	A is unsatisfiable
$A \not\models B$	A doesn't imply B	$A \not\models$	A is satisfiable
$\models A$	A is valid	$A \Leftrightarrow B$	A is equivalent to B
$\not\models A$	A is not valid	$A \models\!\!\!\!\dashv \ B$	“ ” “ ”

<u>Properties of $\models$</u>:

Reflexivity:
$A \models A$

Transitivity:
If $A \models B$ and $B \models C$, then $A \models C$
If $\models A$ and $A \models B$, then $\models B$
If $B \models$ and $A \models B$, then $A \models$

Collapse:
$A_1, \ldots, A_n \models B$ iff $(A_1 \ . \ \cdots \ . \ A_n) \models B$

Contraposition:
$A \models B$ iff $-B \models -A$
$\models A$ iff $-A \models$
$A \models$ iff $\models -A$

Weakening:
If $\models A$, then $B \models A$ (for any B)

Explosion:
If $A \models$, then $A \models B$ (for any B)

<u>Properties of $\Leftrightarrow$</u>:

Reflexivity:
$A \Leftrightarrow A$

Transitivity:
If $A \Leftrightarrow B$ and $B \Leftrightarrow C$, then $A \Leftrightarrow C$

Symmetry:
If $A \Leftrightarrow B$, then $B \Leftrightarrow A$

Contraposition
$A \Leftrightarrow B$ iff $-A \Leftrightarrow -B$

Double Implication:
$A \Leftrightarrow B$ iff both $A \models B$ and $B \models A$

Validity & Unsatisfiability:
If $\models A$, then: $A \Leftrightarrow B$ iff $\models B$
If $A \models$, then: $A \Leftrightarrow B$ iff $B \models$

DEFINITION 1 (SUBSTITUTION)
Suppose S is a schema. Then $S\,[A\;/\;B]$ is the schema we obtain by replacing every instance of the schema A in S with the schema B. This is called **substitution**.

THEOREM 2 (SUBSTITUTING SENTENCE LETTERS WITH SCHEMATA)
Substitution of schemata for sentence letters preserves validity, unsatisfiability, implication, and equivalence. That is, for any schemata A, B, S, and any sentence letter p:

(i) if $\vDash A$, then $\vDash A\,[p\;/\;S]$ (validity preserved)

(ii) if $A \vDash$, then $A\,[p\;/\;S] \vDash$ (unsatisfiability preserved)

(iii) If $A \vDash B$, then $A\,[p\;/\;S] \vDash B\,[p\;/\;S]$ (implication preserved)

(iv) If $A \Leftrightarrow B$, then $A\,[p\;/\;S] \Leftrightarrow B\,[p\;/\;S]$ (equivalence preserved)

WARNING: Satisfiability is *not* preserved under this substitution. For instance, A = "$p\;.\;q$" is satisfiable; but $A\,[p\;/\;-q]$ = "$-q\;.\;q$" is not satisfiable.

THEOREM 3 (SUBSTITUTING SCHEMATA WITH EQUIVALENTS)
The following holds for any schemata A, B, S, T:

(a) Substitution of schemata for equivalent schemata produces equivalent schemata. That is, if $A \Leftrightarrow B$, then $S \Leftrightarrow S\,[A\;/\;B]$.

(b) Substitution of schemata for equivalent scehemata preserves validity, unsatisfiability, implication, equivalence, *and* satisfiability. That is, if $S \Leftrightarrow T$:

  (i) if $\vDash A$, then $\vDash A\,[S\;/\;T]$

  (ii) if $A \vDash$, then $A\,[S\;/\;T] \vDash$

  (iii) if $A \vDash B$, then $A\,[S\;/\;T] \vDash B\,[S\;/\;T]$

  (iv) if $A \Leftrightarrow B$, then $A\,[S\;/\;T] \Leftrightarrow B\,[S\;/\;T]$

  (v) if A is satisfiable, then so is $A\,[S\;/\;T]$

OBSERVATION: One can also do multiple substitutions *simultaneousy*. For instance, we could take "$p\;.\;q$" and replace both sentences letters at the same time with different schemata, e.g. "$(q \vee r)\;.\;(p \supset s)$". Allowing ourselves to do this does not affect the above.