Properties of Implication

Legend			
Symbols	**English**	**Symbols**	**English**
$A \vDash B$	A implies B	$A \vDash$	A is unsatisfiable
$A \nvDash B$	A doesn't imply B	$A \nvDash$	A is satisfiable
$\vDash A$	A is valid	$A \Leftrightarrow B$	A is equivalent to B
$\nvDash A$	A is not valid	$A \dashv\vDash B$	“ ” “ ”

Properties of $\vDash$:

Reflexivity:
$A \vDash A$

Transitivity:
If $A \vDash B$ and $B \vDash C$, then $A \vDash C$
If $\vDash A$ and $A \vDash B$, then $\vDash B$
If $B \vDash$ and $A \vDash B$, then $A \vDash$

Collapse:
$A_1, \ldots, A_n \vDash B$ iff $(A_1 \;.\; \cdots \;.\; A_n) \vDash B$

Contraposition:
$A \vDash B$ iff $-B \vDash -A$
$\vDash A$ iff $-A \vDash$
$A \vDash$ iff $\vDash -A$

Weakening:
If $\vDash A$, then $B \vDash A$ (for any B)

Explosion:
If $A \vDash$, then $A \vDash B$ (for any B)

Properties of $\Leftrightarrow$:

Reflexivity:
$A \Leftrightarrow A$

Transitivity:
If $A \Leftrightarrow B$ and $B \Leftrightarrow C$, then $A \Leftrightarrow C$

Symmetry:
If $A \Leftrightarrow B$, then $B \Leftrightarrow A$

Contraposition
$A \Leftrightarrow B$ iff $-A \Leftrightarrow -B$

Double Implication:
$A \Leftrightarrow B$ iff both $A \vDash B$ and $B \vDash A$

Validity & Unsatisfiability:
If $\vDash A$, then: $A \Leftrightarrow B$ iff $\vDash B$
If $A \vDash$, then: $A \Leftrightarrow B$ iff $B \vDash$

Definition 1 (Substitution)

Suppose S is a schema. Then $S\,[A\ /\ B]$ is the schema we obtain by replacing every instance of the schema A in S with the schema B. This is called **substitution**.

Theorem 2 (Substituting Sentence Letters with Schemata)

Substitution of schemata for sentence letters preserves validity, unsatisfiability, implication, and equivalence. That is, for any schemata A, B, S, and any sentence letter p:

(i) if $\vDash A$, then $\vDash A\,[p\ /\ S]$ (validity preserved)

(ii) if $A \vDash$, then $A\,[p\ /\ S] \vDash$ (unsatisfiability preserved)

(iii) If $A \vDash B$, then $A\,[p\ /\ S] \vDash B\,[p\ /\ S]$ (implication preserved)

(iv) If $A \Leftrightarrow B$, then $A\,[p\ /\ S] \Leftrightarrow B\,[p\ /\ S]$ (equivalence preserved)

WARNING Satisfiability is *not* preserved under this substitution. For instance, A = "$p\ .\ q$" is satisfiable; but $A\,[p\ /\ -q]$ = "$-q\ .\ q$" is not satisfiable.

Theorem 3 (Substituting Schemata with Equivalents)

The following holds for any schemata A, B, S, T:

(a) Substitution of schemata for equivalent schemata produces equivalent schemata. That is, if $A \Leftrightarrow B$, then $S \Leftrightarrow S\,[A\ /\ B]$.

(b) Substitution of schemata for equivalent scehemata preserves validity, unsatisfiability, implication, equivalence, *and* satisfiability. That is, if $S \Leftrightarrow T$:

- (i) if $\vDash A$, then $\vDash A\,[S\ /\ T]$
- (ii) if $A \vDash$, then $A\,[S\ /\ T] \vDash$
- (iii) if $A \vDash B$, then $A\,[S\ /\ T] \vDash B\,[S\ /\ T]$
- (iv) if $A \Leftrightarrow B$, then $A\,[S\ /\ T] \Leftrightarrow B\,[S\ /\ T]$
- (v) if A is satisfiable, then so is $A\,[S\ /\ T]$

✶ **OBSERVATION:** One can also do multiple substitutions *simultaneousy*. For instance, we could take "$p\ .\ q$" and replace both sentences letters at the same time with different schemata, e.g. "$(q \vee r)\ .\ (p \supset s)$". Allowing ourselves to do this does not affect the results above.