Naïve Set Theory

Spring 2013

Abstract

The following is a brief note on naïve set theory. This note contains everything you need to know about sets for Phil 12A, and then some. Section 1 covers the basic definitions and notation regarding sets and membership. Section 2 discusses subsets, intersections, and unions. Section 3 is a "bonus" section, discussing some fun set-theoretic paradoxes.

1 Sets

DEFINITION 1 (SET)
A **set** is a collection of objects that is *uniquely determined by its members*. That is, if A and B are collections with exactly the same members, then A and B are the same set.

EXAMPLE 2
Consider the set of GSIs for Phil 12A. We can denote this set in two ways. One way is to simply list out all of the members of the set, as in:

$$\{\text{Arc, Caitlin, Russ}\}$$

When listing the members of set, neither order nor repetitions matter. So this set is identical to both of these sets:

$$\{\text{Russ, Caitlin, Arc}\}\,,\ \{\text{Russ, Caitlin, Arc, Caitlin, Russ}\}$$

Another way to denote the same set is to give a *description* that all and only the members of the set satisify. So we can also write the set of GSIs for Phil 12A as:

$$\{x \mid x \text{ is a GSI for Phil 12A}\}$$

EXAMPLE 3

When writing a set in this second way, we're allowed to use any description we want to the right of the "|" symbol. So, for instance, we can write:

$$\{x \mid x \text{ is a dog}\}$$
$$\{x \mid x \text{ is a student in Phil 12A}\}$$
$$\{n \mid n \text{ is a negative number}\}$$
$$\{j \mid j \text{ is a jedi}\}$$
$$\{s \mid s \text{ is a set of dogs}\}$$

OBSERVATION:

(1) Sets can be infinite in size, as this third set shows.

(2) Sets can be empty, as this fourth set shows: since there are no jedi, nothing is a member of $\{j \mid j \text{ is a jedi}\}$. We call such a set the **empty set**, which we denote by "$\varnothing$".

(3) Sets can have *other sets* as members, as this fifth set shows. After all, sets are things too. (Or are they?)

DEFINITION 4 (MEMBERSHIP)

When an object a is a **member** of a set A, we write "$a \in A$." If a is not a member of A, we write "$a \notin A$."

Example 5
Some examples of membership and lack thereof (check them yourself):

$$\begin{aligned}
\text{Arc} &\in \{x \mid x \text{ is a GSI for Phil 12A}\} \\
2 &\in \{1,2,3\} \\
\text{Russ} &\in \{\text{Bertrand Russell, Russell Crowe, Russell Buehler}\} \\
\varnothing &\in \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \ldots\} \\
6 &\notin \{\text{Russ, Caitlin, Arc}\} \\
4 &\notin \{n \mid n \text{ is a prime number}\} \\
\text{“June”} &\notin \{w \mid w \text{ is an acceptable word in Scrabble}\} \\
\{1,2\} &\notin \{1,2,3\}
\end{aligned}$$

2 Subsets, Intersections, Unions

Definition 6 (Subset)
We say that a set A is a **subset** of a set B if every member of A is a member of B. If A is a subset of B, we write “$A \subseteq B$.”

We can picture subsets in the following way:

$A \subseteq B$: $A \nsubseteq B$:

Here, “A is a subset of B” means that anything inside A is also inside B. This is true in the left case, but not in the right case.

EXAMPLE 7
Some examples of subsets and non-subsets:

$$\{1,2\} \subseteq \{1,2,3\}$$
$$\{\text{Frank, Dino, Nat}\} \subseteq \{x \mid x \text{ is a great singer}\}$$
$$\{\text{Frank, Dino}\} \nsubseteq \{\text{Frank, Nat, Louis}\}$$
$$\{n \mid n \text{ is even}\} \nsubseteq \{n \mid n \text{ is odd}\}$$

OBSERVATION: A few of points about subsets:

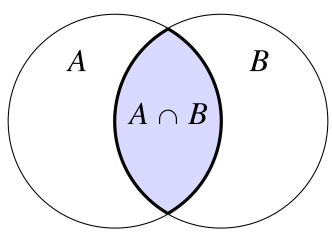
(1) Both A and $\varnothing$ are automatically subsets of A.

(2) The subset relation is transitive, i.e. if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Note that this *is not* the case for membership: for instance, $1 \in \{1,2\}$, and $\{1,2\} \in \{\{1,2\},\{3\}\}$, but $1 \notin \{\{1,2\},\{3\}\}$.

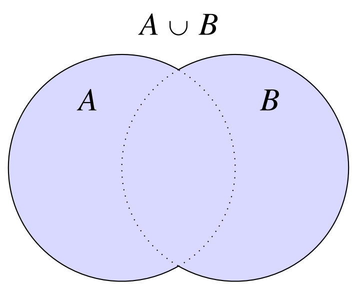
(3) If A has n elements, then A has 2^n subsets.[a]

[a] For math aficionados, you can prove this in the finite case using the binomial theorem. Try it!

DEFINITION 8 (INTERSECTION)
The **intersection** of A and B is the set of all elements that are in both A and B. The intersection of A and B is written as "$A \cap B$". In a picture:



DEFINITION 9 (UNION)
The **union** of A and B is the set of elements that are in either A or B. The union of A and B is written "$A \cup B$". In a picture:



EXAMPLE 10
Some examples of intersections and unions:

$$\{1,2\} \cup \{3,4\} = \{1,2,3,4\}$$
$$\{1,2\} \cup \{2,3\} = \{1,2,3\}$$
$$\{1,2\} \cap \{2,3\} = \{2\}$$
$$\{1,2\} \cap \{3,4\} = \varnothing$$

OBSERVATION: Some useful facts about intersections and unions:

(1) For any A, $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.

(2) For any A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(3) For any A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

(4) For any A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3 Bonus: Two Paradoxes

3.1 Galileo's Paradox

Which of these sets is bigger?

$$\begin{aligned}\mathbb{N} &= \{n \mid n \text{ is a (non-negative) integer}\} \\ &= \{0, 1, 2, 3, 4, 5, \ldots\} \\ \mathbb{E} &= \{n \mid n \text{ is an even integer}\} \\ &= \{0, 2, 4, 6, 8, 10, \ldots\}\end{aligned}$$

Many people immediately say, "$\mathbb{N}$, of course! After all, $\mathbb{E}$ is a subset of $\mathbb{N}$. $\mathbb{N}$ has all of the numbers in $\mathbb{E}$ and then some!" However, one could just as easily say, "Look, both $\mathbb{N}$ and $\mathbb{E}$ are infinite sets. Therefore, both sets have the same size: they're both infinitely big!"

So we have two arguments. One says that $\mathbb{N}$ is bigger than $\mathbb{E}$. The other says $\mathbb{N}$ and $\mathbb{E}$ have the same size. So which is right?

This paradox was pointed out by Galileo in 1638 with his final work, *Two New Sciences*. In that work, Galileo uses this paradox to show that we cannot meaningfully compare the "sizes" of infinite totalities. Hence, we cannot meaningfully compare the size of $\mathbb{N}$ and $\mathbb{E}$. In case you want to read what he said:[1]

> SALV. . . . if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?[2]
>
> SIMP. Most certainly.
>
> SIMP. If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.
>
> SIMP. Precisely so.
>
> SALV. But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root

[1] Galilei, Galileo. *Discourses and Mathematical Demonstrations Realting to Two New Sciences*, pp. 1-61. 1638. Web. http://galileoandeinstein.physics.virginia.edu/tns_draft/tns_001to061.html

[2] By "square", he means a perfect square, e.g. 1, 4, 9, 16, 25, etc.

> of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares...
>
> SAGR. What then must one conclude under these circumstances?
>
> SALV. So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; **neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities.**

So that's it then. The sizes of $\mathbb{N}$ and $\mathbb{E}$ are incomparable. Or are they?

A German mathematician, Georg Cantor, thought otherwise. Cantor argued that one *could* meaningfully compare the sizes of $\mathbb{N}$ and $\mathbb{E}$ by using the notion of a "one-to-one correspondence." The idea is captured in the following principle:

Conjecture 11 (One-to-One Correspondence)
Two sets A and B have the same size iff one can find a way to pair each element of A with exactly one element from B and *vice versa*, i.e. if one can find a "one-to-one correspondence" between the elements of A and the elements of B.

Take an example: suppose you want to know whether a box of utensils contains the same number of forks as knives. You could just count the forks, then count the knives, and then compare the two numbers. But another way to do it is to set a table, placing exactly one fork and one knife on any given placemate. Then you'd know if you had the same number of forks and knives if every placemate had exactly one fork and one knife.

The idea is the same for sets in general. To find out if A and B have the same number of elements, put an element of A and an element of B on every "placemate," and see if there are any placemates with just one element. If there is, then A and B have different sizes. If there isn't, then A and B have the same size.

So what about $\mathbb{N}$ and $\mathbb{E}$? Well, one way to "set the table" is as follows:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \dots & n & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow & \dots \\ 2 & 4 & 6 & 8 & \dots & 2n & \dots \end{array}$$

Every element of $\mathbb{N}$ corresponds to exactly one element in $\mathbb{E}$, and similarly every element of $\mathbb{E}$ corresponds to exactly one element in $\mathbb{N}$. So by the one-to-one correspondence principle, we can conclude that $\mathbb{N}$ and $\mathbb{E}$ have the same size.

What about $\mathbb{N}$ and

$$\begin{aligned} \mathbb{Z} &= \{n \mid n \text{ is an integer}\} \\ &= \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \end{aligned}$$

Is $\mathbb{Z}$ bigger than $\mathbb{N}$, or does it have the same size?

It turns out $\mathbb{N}$ and $\mathbb{Z}$ *do* have the same size. For instance, we can set up a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$ as follows:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ 1 & -1 & 2 & -2 & 3 & -3 & \dots \end{array}$$

That is, we have the n^{th} odd number correspond to the n^{th} positive number, and the n^{th} even number correspond to the n^{th} negative number.

We can also find a one-to-one correspondence between $\mathbb{N}$ and:

$$\mathbb{Q}^{+} = \{n \mid n \text{ is a positive fraction}\}$$

Hence, $\mathbb{N}$ and $\mathbb{Q}^{+}$ have the same size.

"Okay, but so what?" you say. "Why is this interesting? So far, it looks like all this principle tells us is that infinite sets are all the same size, viz. they're infinite. That's not very interesting. We already knew that!"

Well, actually. . .

Theorem 12 (Cantor's Diagonalization Argument)
The set

$$\mathbb{R} = \{n \mid n \text{ is a real number}\}$$

is *strictly bigger* than $\mathbb{N}$. That is, $\mathbb{N}$ has a *smaller* size than $\mathbb{R}$. In fact, there are more real numbers between 0 and 1 than there are numbers in $\mathbb{N}$!

▶ **Proof:** Suppose not, for *reductio*. Then, according to our principle, there must be a one-to-one correspondence between $\mathbb{N}$ and the interval $[0, 1]$. Since every number between 0 and 1 can be written as a decimal, that correspondence will look something like:

$$\begin{aligned}
1 &\to 0.7369205810582\ldots \\
2 &\to 0.3141592653589\ldots \\
3 &\to 0.2222222222222\ldots \\
4 &\to 0.9999000000000\ldots \\
5 &\to 0.1000000000000\ldots \\
&\vdots \\
n &\to 0.n_1 n_2 n_3 n_4 n_5 n_6 n_7 n_8 \ldots \\
&\vdots
\end{aligned}$$

It may not be this exact correspondence, but the particular numbers won't matter in what follows: we just need *some* such correspondence to exist.

Given this correspondence, we will construct a new real number r such that: (i) $0 \leqslant r \leqslant 1$ (ii) r does not correspond to any number on this list. If r does not correspond to any number on this list, then our proposed correspondence cannot really be a *one-to-one* correspondence. And since this correspondence was arbitrary, it follows that *there is no one-to-one correspondence* between $\mathbb{N}$ and $[0, 1]$, contradicting our original assumption.

So now we must construct r in such a way so that these two properties hold. We proceed as follows: for each $n \in \mathbb{N}$ on the lefthand side of this list, take the n^{th} digit of the real number (between 0 and 1) that corresponds to

n, and add 1 to that digit (if it's 9, make it go to 0); then set that new digit as r's n^{th} digit. So for instance, in the correspondence above:

$$
\begin{aligned}
1 &\to 0.\boxed{7}369205810582\ldots \\
2 &\to 0.3\boxed{1}41592653589\ldots \\
3 &\to 0.22\boxed{2}2222222222\ldots \\
4 &\to 0.999\boxed{9}000000000\ldots \\
5 &\to 0.1000\boxed{0}00000000\ldots \\
&\vdots \\
n &\to 0.n_1n_2n_3n_4n_5n_6n_7n_8\ldots\boxed{n_n}\ldots \\
&\vdots
\end{aligned}
$$

So our r would be $0.82301\ldots$.

I claim that r is nowhere to be found in this list. Why? Well, suppose (for *reductio*) it was; say that the number $d \in \mathbb{N}$ corresponded to r. Question: what is the d^{th} digit of r? Well, according to our rule for constructing r, the d^{th} digit of r will be the d^{th} digit of the number d corresponds to plus 1. But d corresponds to r. So the d^{th} digit of r will be the d^{th} of r plus 1. But that's a contradiction: no number can be equal to itself plus 1!

Hence, r is not on this list. And if r isn't on this list, i.e. if r doesn't correspond to any number on the left, then our "correspondence" was not a one-to-one correspondence after all. So no matter what correspondence we propose, there will *always* be numbers we left out. □

Hence $\mathbb{R}$ is strictly bigger than $\mathbb{N}$. But aren't both sets infinite? How could one be bigger than another?

Corollary 13
There are *different sizes of infinity*!

Comment: How many sizes of infinity? *Too many*. For each infinity, there's another infinity bigger than it. But you can't just say that there are an infinite number of infinities. If I asked, "How many infinities are there?" and you replied, "Infinitely many," I could reply, "What size of infinity?".

3.2 Russell's Paradox

Consider the following set:

$$R = \{s \mid s \notin s\}$$

Is $R \in R$?

Suppose $R \in R$. Since for any set s, $s \in R$ implies that $s \notin s$, it follows that $R \notin R$. So if $R \in R$, then $R \notin R$.

So suppose $R \notin R$. Since for any s, $s \notin s$ implies that $s \in R$, it follows that $R \in R$. So if $R \notin R$, then $R \in R$.

So $R \in R$ iff $R \notin R$. But that's like saying a sentence of the form "A iff $-A$" is true. And *no* sentence of that form can be true. And yet the arguments above both look completely legitimate. So what's going on?

One very natural response to have is to point out that there's something very fishy with the idea that a set can contain itself. After all, if sets could contain themselves, then we could define a set S such that $S = \{S\}$. Then:

$$S = \{S\} = \{\{S\}\} = \{\{\{S\}\}\} = \dots$$

In such a situation, we might feel queasy because we could never explicitly write out what the elements of S really were. S, in some sense, seems *unfounded*.

However, as it stands, such a response doesn't quite suffice. If you just say that you're going to rule out sets being members of themselves, then R will be the set of *absolutely everything*. But then R will contain every set, including itself (which is what we wanted to avoid). You need to say more about how to get rid of self-containing sets without falling into inconsistency.

Bertrand Russell pointed out this paradox in 1901 (though Ernst Zermelo had already known about the paradox a year earlier). The problem can be compared to the following paradox:

> Suppose there is a barber in Berkeley that shaves all and only those men who don't shave themselves. This sounds innocent enough (after all, isn't that what barbers *do*, i.e. shave people who don't do it themselves?). But does the barber shave himself? If he does, then since he only shaves men who don't shave themselves, the barber must not shave himself. But if he doesn't, then since the barber shaves every man who doesn't shave himself, it follows that the barber shaves himself. Paradox.

The most natural way to resolve the "barbershop paradox" is to simply argue that what the paradox shows is that there can be no such barber. The existence of such a barber is inconsistent. This can be seen by showing that the following first-order schema is inconsistent:

$$\forall x \ (\text{Shaves}(\text{barber}, x) \equiv -\text{Shaves}\,(x, x))$$

This can be shown to be unsatisfiable since you can always have x pick out the barber.

But in the case of sets as we've defined them, we have no way to rule out the existence of such a set. Naïve set theory is in fact inconsistent, and thus it entails absolutely everything. So if we want a *consistent* theory of sets, we need to be "less naïve" and modify our appraoch. This is often done through axiomatizations such as ZFC, which you can learn more about in nearly any introduction to set theory. But for the purposes of this course, we can ignore this inconsistency, and simply avoid the use of such problematic sets. (We'll mostly only be talking about sets of things like dogs, people, mammals, etc., which are harmless from this perspective.)