Sets, Relations, and Functions

Phil 143 Handout

§1 Basic Definitions and Notation

Sets are denoted with curly braces, $\{\ldots\}$. Examples:

- $\{a, b, c\}$
- $\{1, 2, 6, 108\}$
- $\{\text{Arc}, \text{Wes}, 4, \text{the color blue}\}$
- $\{a, b, \{a, b\}\}$
- $\{n \mid n \text{ is an even number}\} = \{0, 2, 4, 6, \ldots\}$

Read '$\{x \mid \phi(x)\}$' as denoting the set of all x such that $\phi(x)$. Notice that by the fourth example, sets can be memebers of other sets.

Ordered pairs are denoted with angle brackets, $\langle a, b\rangle$. In general, ordered n-tuples are written as $\langle a_1, \ldots, a_n\rangle$.

- For sets, neither order nor repetition matter. So e.g., $\{a, a, b\} = \{a, b\} = \{b, a\}$.
- For orderd n-tuples, both order and repetition matter. So $\langle a, a, b\rangle \neq \langle a, b\rangle \neq \langle b, a\rangle$.

Concept	**Symbol**	**Meaning**
x is in A	$x \in A$	x is a member of A
A and B are the same set	$A = B$	$\forall x\colon x \in A$ iff $x \in B$
A is a subset of B	$A \subseteq B$	$\forall x\colon$ if $x \in A$, then $x \in B$
A is a proper subset of B	$A \subsetneq$	$A \subseteq B$ and $A \neq B$
A is a superset of B	$A \supseteq B$	$B \subseteq A$
The empty set	$\varnothing$	the set with no members, $\{\}$
The power set of A	$\wp A$	the set of all subsets of A, $\{B \mid B \subseteq A\}$
The union of A and B	$A \cup B$	$\{x \mid x \in A \text{ or } x \in B\}$
The intersection of A and B	$A \cap B$	$\{x \mid x \in A \text{ and } x \in B\}$
The complement of A relative to B	$B - A$ or A^c	$\{x \mid x \in B \text{ and } x \notin A\}$
The product of A and B	$A \times B$	$\{\langle a, b\rangle \mid a \in A \text{ and } b \in B\}$
The n^{th} power of A	A^n	$A \times A \times \cdots \times A$ (n times)
R is a binary relation on A and B		$R \subseteq A \times B$
a is R-related to b	Rab or aRb	$\langle a, b\rangle \in R$
The domain of R	$\mathsf{dom}\,(R)$	$\{a \mid \exists b\colon\ \langle a, b\rangle \in R\}$
The range of R	$\mathsf{ran}\,(R)$	$\{b \mid \exists a\colon\ \langle a, b\rangle \in R\}$
f maps A to B	$f\colon A \to B$	$f \subseteq A \times B$ is a function
The image of f	$\mathsf{im}\,(f)$	$\{b \in B \mid \exists a \in A\colon\ f(a) = b\}$
f is injective (one-to-one)		$\forall x, y \in A\colon\ f(x) = f(y) \Rightarrow x = y$
f is surjective (onto)		$\forall x \in B\, \exists x \in A\colon\ f(x) = y$
f is bijective		f is injective and surjective

§2 Counting

✎ **Notation:** If A is a set, then $|A|$ will be the size of A (i.e., how many elements are in A). So for instance, if $|A| = 4$, that means that A has 4 distinct members.

Also, recall: $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

Here are some general principles that will be useful for the problem set (we'll derive them in section, but no need to know how they're derived):

Fact (*Essential Counting*). Let A and B be sets throughout.

- $|\wp A| = 2^{|A|}$. That is, A has $2^{|A|}$ distinct subsets.

  Example: if $|A| = 4$ (that is, if A has 4 elements), then A has $2^4 = 16$ distinct subsets.

- $|A \times B| = |A| \times |B|$. That is, the size of a product is just the product of the sizes.

  Example: If $|A| = 5$ and $|B| = 7$, $|A \times B| = 5 \times 7 = 35$.

  In particular, the number of binary relations on A and B is just the number of distinct subsets of $A \times B$. Hence, the number of binary relations on $A \times B$ is $2^{|A \times B|} = 2^{35} \approx 3.44 \cdot 10^{10}$ (that's a lot...)

- $|B^A| = |B|^{|A|}$. That is, the number of distinct functions from A to B is $|B|^{|A|}$.

  Example: If $|A| = 4$ and $|B| = 3$, then the number of functions from A to B is $3^4 = 81$.

For counting the number of injective and surjective functions, if the size of A and B is small, you're better off just listing them out and counting by hand. But here's how to count them, in case you're curious:

Fact (*Inessential Counting*).

- If $|A| \leqslant |B|$, the number of injective functions from A to B is given by:

  $$\frac{|B|!}{(|B| - |A|)!}$$

  Otherwise, there are no injective functions from A to B.

- If $|A| \geqslant |B|$, the number of surjective functions is given by:

  $$\sum_{k=0}^{|B|} (-1)^k \binom{|B|}{k} (|B| - k)^{|A|}$$

  Otherwise, there are no surjective functions.

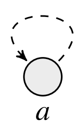
- If $|A| = |B|$, the number of bijective functions is $|A|!$. Otherwise, there are no bijective functions from A to B.

§3 Properties of Relations

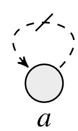
We'll mostly focus on binary relations. Let $R \subseteq A^2$ be a binary relation on A. We'll abbreviate '$\langle a, b \rangle \in R$' as '$Rab$', and '$\langle a, b \rangle \notin R$' as '$\sim Rab$'.

- R is ***reflexive*** if for all $a \in A$, Raa.
- R is ***irreflexive*** if for all $a \in A$, $\sim Raa$.
- R is ***symmetric*** if for all $a, b \in A$, if Rab, then Rba.
- R is ***asymmetric*** if for all $a, b \in A$, if Rab, then $\sim Rba$.
- R is ***anti-symmetric*** if for all $a, b \in A$, if Rab and Rba, then $a = b$.
- R is ***transitive*** if for all $a, b, c \in A$, if Rab and Rbc, then Rac.
- R is ***intransitive*** if for all $a, b, c \in A$, if Rab and Rbc, then $\sim Rac$.
- R is ***connected*** if for all $a, b \in A$, if $a \neq b$, then either Rab or Rba.

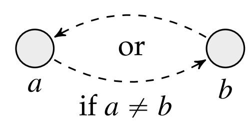
Here are the properties represented in a diagram. Think of an arrow from a to b as saying that $\langle a, b \rangle \in R$, and the lack of an arrow from a to b as saying that $\langle a, b \rangle \notin R$. The presence of the dashed arrows follows from the presence of the solid arrows, given the corresponding property holds.



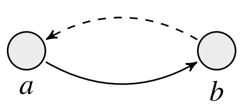
Reflexivity



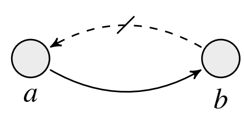
Irreflexivity



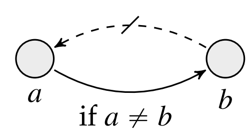
Connectedness



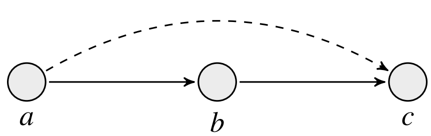
Symmetry



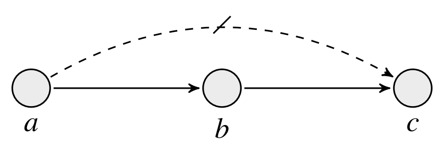
Asymmetry



Anti-symmetry



Transitivity



Intransitivity