Axiomatic Proofs

Phil 143 Worksheet

1. Produce a formal proof in **K** for the following formulas:

(a) $\Box p \to \Box(q \to p)$

▶ Solution:

1.	$p \to (q \to p)$	(taut)
2.	$\Box(p \to (q \to p))$	(Nec, 1)
3.	$\Box(p \to (q \to p)) \to (\Box p \to \Box(q \to p))$	(K)
4.	$\Box p \to \Box(q \to p)$.	(MP, 3, 4)

(b) $(\Diamond p \to \Box q) \to (\Box p \to \Box q)$

▶ Solution:

1.	$\neg p \to (p \to q)$	(taut)
2.	$\Box(\neg p \to (p \to q))$	(Nec, 1)
3.	$\Box(\neg p \to (p \to q)) \to (\Box\neg p \to \Box(p \to q))$	(K)
4.	$\underline{\Box\neg p}_A \to \underline{\Box(p \to q)}_B$	(MP, 3, 4)
5.	$\underline{\Box(p \to q)}_B \to \underline{(\Box p \to \Box q)}_C$	(K)
6.	$(A \to B) \to ((B \to C) \to (A \to C))$	(taut)
7.	$(B \to C) \to (A \to C)$	(MP, 4, 6)
8.	$A \to C$ $= \Box\neg p \to (\Box p \to \Box q)$	(MP, 5, 7)
9.	$\underline{\Box q}_D \to \underline{(\Box p \to \Box q)}_C$ (taut)	
10.	$(A \to C) \to ((D \to C) \to ((A \vee D) \to C))$	(taut)
11.	$(D \to C) \to ((A \vee D) \to C)$	(MP, 8, 10)
12.	$(A \vee D) \to C$ $= (\Box\neg p \vee \Box q) \to (\Box p \to \Box q)$	(MP, 9, 11)
13.	$((A \vee D) \to C) \to ((\neg A \to D) \to C)$	(taut)
14.	$(\neg A \to D) \to C$ $= (\neg\Box\neg p \to \Box q) \to (\Box p \to \Box q)$ $= (\Diamond p \to \Box q) \to (\Box p \to \Box q)$.	(MP, 12, 13)

(c) $\Box(p \to q) \to (\Diamond p \to \Diamond q)$

▶ Solution:

1. $(p \to q) \to (\neg q \to \neg p)$ (taut)
2. $\Box((p \to q) \to (\neg q \to \neg p))$ (Nec, 1)
3. $\Box((p \to q) \to (\neg q \to \neg p)) \to (\Box(p \to q) \to \Box(\neg q \to \neg p))$ (K)
4. $\underline{\Box(p \to q)}_A \to \underline{\Box(\neg q \to \neg p)}_B$ (MP, 2, 3)
5. $\underline{\Box(\neg q \to \neg p)}_B \to \underline{(\Box \neg q \to \Box \neg p)}_C$ (K)
6. $\underline{(\Box \neg q \to \Box \neg p)}_C \to \underline{(\neg \Box \neg p \to \neg \Box \neg q)}_D$ (taut)
7. $(A \to B) \to ((B \to C) \to ((C \to D) \to (A \to D)))$ (taut)
8. $(B \to C) \to ((C \to D) \to (A \to D))$ (MP, 4, 7)
9. $(C \to D) \to (A \to D)$ (MP, 5, 8)
10. $A \to D$ (MP, 6, 9)
    $= \Box(p \to q) \to (\neg \Box \neg p \to \neg \Box \neg q)$
    $= \Box(p \to q) \to (\Diamond p \to \Diamond q)$.

(d) $\Diamond(p \to q) \to (\Box p \to \Diamond q)$

▶ Solution:

1. $p \to (\neg q \to \neg(p \to q))$ (taut)
2. $\Box(p \to (\neg q \to \neg(p \to q)))$ (Nec, 1)
3. $\Box(p \to (\neg q \to \neg(p \to q))) \to (\Box p \to \Box(\neg q \to \neg(p \to q)))$ (K)
4. $\underline{\Box p}_A \to \underline{\Box(\neg q \to \neg(p \to q))}_B$ (MP, 3, 4)
5. $\underline{\Box(\neg q \to \neg(p \to q))}_B \to \underline{(\Box \neg q \to \Box \neg(p \to q))}_C$ (K)
6. $\underline{(\Box \neg q \to \Box \neg(p \to q))}_C \to \underline{(\neg \Box \neg(p \to q) \to \neg \Box \neg q)}_D$ (taut)
7. $(A \to B) \to ((B \to C) \to ((C \to D) \to (A \to D)))$ (taut)
8. $(B \to C) \to ((C \to D) \to (A \to D))$ (MP, 4, 7)
9. $(C \to D) \to (A \to D)$ (MP, 5, 8)
10. $A \to D$ (MP, 6, 9)
    $= \Box p \to (\underline{\neg \Box \neg(p \to q)}_E \to \underline{\neg \Box \neg q}_F)$
11. $(A \to (E \to F)) \to (E \to (A \to F))$ (taut)
12. $E \to (A \to F)$ (MP, 10, 11)
    $= \neg \Box \neg(p \to q) \to (\Box p \to \neg \Box \neg q)$
    $= \Diamond(p \to q) \to (\Box p \to \Diamond q)$.

2. Show that the following rules are derivable in **K**:

(a) If $\vdash_{\mathbf{K}} (\varphi_1 \wedge \varphi_2) \to \psi$, then $\vdash_{\mathbf{K}} (\Box\varphi_1 \wedge \Box\varphi_2) \to \Box\psi$.

► Solution: Suppose $\vdash_{\mathbf{K}} (\varphi_1 \wedge \varphi_2) \to \psi$. Then by Necessitation, $\vdash_{\mathbf{K}} \Box((\varphi_1 \wedge \varphi_2) \to \psi)$. Using the K axiom and Modus Ponens, $\vdash_{\mathbf{K}} \Box(\varphi_1 \wedge \varphi_2) \to \Box\psi$. But we know that $\vdash_{\mathbf{K}} (\Box\varphi_1 \wedge \Box\varphi_2) \to \Box(\varphi_1 \wedge \varphi_2)$. So using 12A reasoning, $\vdash_{\mathbf{K}} (\Box\varphi_1 \wedge \Box\varphi_2) \to \Box\psi$.

(b) If $\vdash_{\mathbf{K}} \varphi \to (\psi_1 \vee \psi_2)$, then $\vdash_{\mathbf{K}} \Diamond\varphi \to (\Diamond\psi_1 \vee \Diamond\psi_2)$.

► Solution: Suppose $\vdash_{\mathbf{K}} \varphi \to (\psi_1 \vee \psi_2)$. By 12A reasoning, this is equivalent to $\vdash_{\mathbf{K}} (\neg\psi_1 \wedge \neg\psi_2) \to \neg\varphi$. But by the above fact, we can infer that $\vdash_{\mathbf{K}} (\Box\neg\psi_1 \wedge \Box\neg\psi_2) \to \Box\neg\varphi$. By 12A reasoning again, this is equivalent to $\vdash_{\mathbf{K}} \neg\Box\neg\varphi \to (\neg\Box\neg\psi_1 \vee \neg\Box\neg\psi_2$. Replacing $\neg\Box\neg$ with $\Diamond$, we get that $\vdash_{\mathbf{K}} \Diamond\varphi \to (\Diamond\psi_1 \vee \Diamond\psi_2)$.

3. Suppose Γ is a maximal **K**-consistent set. Prove the following:

(a) If $\varphi \vee \psi \in \Gamma$, then either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

► Solution: Suppose for *reductio* that $\varphi \vee \psi \in \Gamma$, but that $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is maximal, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. But since $((\varphi \vee \psi) \wedge \neg\varphi \wedge \neg\psi) \to \bot$ is a propositional tautology, it follows that $\Gamma \vdash_{\mathbf{K}} \bot$. But this can't be, since Γ is **K**-consistent. Hence, contrary to our original supposition, either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

(b) If $\Box\varphi \in \Gamma$ and $\Box\psi \in \Gamma$, then $\Box(\varphi \wedge \psi) \in \Gamma$.

► Solution: Suppose for *reductio* that $\Box\varphi \in \Gamma$ and $\Box\psi \in \Gamma$, but that $\Box(\varphi \wedge \psi) \notin \Gamma$. Since Γ is maximal, $\neg\Box(\varphi \wedge \psi) \in \Gamma$. However, here's a proof in **K** that shows that $(\Box\varphi \wedge \Box\psi \wedge \neg\Box(\varphi \wedge \psi)) \to \bot$ is **K**-provable:

1.	$\varphi \to (\psi \to (\varphi \wedge \psi))$	(taut)
2.	$\Box(\varphi \to (\psi \to (\varphi \wedge \psi)))$	(Nec, 1)
3.	$\Box(\varphi \to (\psi \to (\varphi \wedge \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi)))$	(K)
4.	$\underline{\Box\varphi}_A \to \underline{\Box(\psi \to (\varphi \wedge \psi))}_B$	(MP, 3, 4)
5.	$\underline{\Box(\psi \to (\varphi \wedge \psi))}_B \to \underline{(\Box\psi \to \Box(\varphi \wedge \psi))}_C$	(K)
6.	$(A \to B) \to ((B \to C) \to (A \to C))$	(taut)
7.	$(B \to C) \to (A \to C)$	(MP, 4, 6)
8.	$A \to C$ $= \Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))$	(MP, 5, 7)
9.	$(\Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))) \to ((\Box\varphi \wedge \Box\psi \wedge \neg\Box(\varphi \wedge \psi)) \to \bot)$	(taut)
10.	$(\Box\varphi \wedge \Box\psi \wedge \neg\Box(\varphi \wedge \psi)) \to \bot$	(MP, 8, 9)

Hence, $\Gamma \vdash_{\mathbf{K}} \bot$. But this can't be, since Γ is **K**-consistent. Hence, contrary to our original supposition, $\Box(\varphi \wedge \psi) \in \Gamma$.