Knowledge

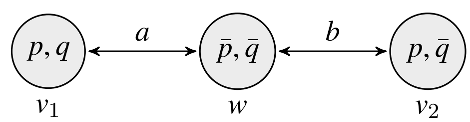
Phil 143 Worksheet

Let $A \subseteq \mathsf{Agt}$ be a set of agent symbols. For each of the formulas below, determine whether or not that formula is valid over symmetric epistemic frames. If it is valid over symmetric epistemic frames, prove it. If not, construct a symmetric epistemic model that falsifies the formula.

1. $\neg\varphi \to \mathsf{E}_A \neg \mathsf{E}_A \varphi$

> ▶ Solution: Valid. Suppose $\mathcal{M}, w \vDash \neg\varphi$. By symmetry, we know that for each agent $a \in A$, $\mathcal{M}, w \vDash \mathsf{K}_a \neg \mathsf{K}_a \varphi$—that is, a knows that they themselves don't know φ. Hence, for each agent a, $\mathcal{M}, w \vDash \mathsf{K}_a \neg \mathsf{E}_A \varphi$—that is, a knows that not everyone knows φ. And since this holds for all agents, it follows that $\mathcal{M}, w \vDash \mathsf{E}_A \neg \mathsf{E}_A \varphi$.

2. $\neg\varphi \to \mathsf{S}_A \neg \mathsf{S}_A \varphi$

> ▶ Solution: Not valid. Consider the model below.
>
> 
>
>
> $\mathcal{M}, w \vDash \neg p$. But $\mathcal{M}, w \vDash \neg \mathsf{K}_a \neg \mathsf{S}_A p$ because there is a $v \in W$ such that $wR_a v$ (namely v_1) such that $\mathcal{M}, v \vDash \mathsf{K}_b p$ (and hence $\mathcal{M}, v \vDash \mathsf{S}_A p$), and there is a $u \in W$ such that $wR_b u$ (namely v_2) such that $\mathcal{M}, u \vDash \mathsf{K}_a p$ (and hence $\mathcal{M}, u \vDash \mathsf{S}_A p$). So no agent in A knows that no one knows p, i.e., $\mathcal{M}, w \nvDash \mathsf{S}_A \neg \mathsf{S}_A p$.

3. $\neg\varphi \to \mathsf{D}_A \neg \mathsf{D}_A \varphi$

> ▶ Solution: Valid. Suppose $\mathcal{M}, w \vDash \neg\varphi$, where each R_a is symmetric. Remember, $\mathcal{M}, v \vDash \mathsf{D}_A \varphi$ iff for all $u \in \bigcap_{a \in A} R_a(v)$, $\mathcal{M}, u \vDash \varphi$. So let $v \in \bigcap_{a \in A} R_a(w)$ (i.e., v is such that for all $a \in A$, $wR_a v$). Since each R_a is symmetric, it follows that for all $a \in A$, $vR_a w$, i.e., $w \in \bigcap_{a \in A} R_a(v)$. So there is a $u \in \bigcap_{a \in A} R_a(v)$ such that $\mathcal{M}, u \nvDash \varphi$. Hence, $\mathcal{M}, v \vDash \neg \mathsf{D}_A \varphi$. And since this holds for all $v \in \bigcap_{a \in A} R_a(w)$, it follows that $\mathcal{M}, w \vDash \mathsf{D}_A \neg \mathsf{D}_A \varphi$.

4. $\neg\varphi \to \mathsf{C}_A \neg \mathsf{C}_A \varphi$

► Solution: Valid. Suppose $\mathcal{M}, w \models \neg\varphi$, where each R_a is symmetric. Suppose there's a finite path $v_1, \ldots, v_n$ such that for some $a_1, \ldots, a_n \in A$, $wR_{a_1}v_1$ and $v_1R_{a_2}v_2$ and ... and $v_{n-1}R_{a_n}v_n$. By symmetry, it follows that $v_nR_{a_n}v_{n-1}$ and ... and $v_2R_{a_2}v_1$ and $v_1R_{a_1}w$. Hence, for every finite path from w, there's a finite path back to w. And since $\mathcal{M}, w \not\models \varphi$, it follows that for every finite away from w, there's a finite path to a state where $\neg\varphi$ is true, i.e., $\mathcal{M}, w \models \mathsf{C}_A \neg \mathsf{C}_A\varphi$.