INTUITIONISTIC LOGIC

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APRIL 13–15TH, 2015

§1 BRANCHES OF LOGIC

1. Mathematical Logic
   (a) model theory
   (b) computability theory
   (c) set theory
   (d) proof theory
2. Philosophical Logic
   (a) modal logic
   (b) nonclassical logics
   (c) higer-order logics
   (d) formal epistemology
   (e) paradoxes
3. Philosophy of Logic
   (a) logical pluralism
   (b) logical constants
   (c) normativity of logic
   (d) theories of truth

§2 NONCLASSICAL LOGICS

1. Paracomplete logic: $\nvDash \varphi \vee \neg\varphi$
   - intuitionistic logic: $\nvDash \neg\neg\varphi \to \varphi$
   - many-valued logic: no bivalence
2. Paraconsistent logic: $\nvDash \neg(\varphi \wedge \neg\varphi)$
   - relevant logic: $\varphi \vee \psi, \neg\varphi \nvDash \psi$
   - linear logic: relevant + intuitionistic − contraction
3. Quantum logic: $\varphi \wedge (\psi \vee \theta) \nvDash (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$

§3 MOTIVATION FOR INTUITIONISTIC LOGIC

1. **Mathematics**: Suppose the twin primes conjecture is such that the only way it could be settled was by brute force checking. Would the twin primes conjecture be true or false?

   Some think neither: mathematical statements are not just true or false independently of our ability to prove them. Mathematical reality does not sit out there for us to discover, but is rather something we construct. Thus, mathematical statements are neither true nor false until we've constructed proofs (or refutations) of them.

   What would it take to prove a logically complex mathematical statement?

$$
\begin{array}{lcl}
x \Vdash \varphi \wedge \psi & \Leftrightarrow & x = \langle y, z\rangle \text{ where } y \Vdash \varphi \text{ and } z \vdash \psi \\
x \Vdash \varphi \vee \psi & \Leftrightarrow & x \Vdash \varphi \text{ or } x \Vdash \psi \\
x \Vdash \neg\varphi & \Leftrightarrow & \forall y\colon\ y \Vdash \varphi \Rightarrow x(y) \Vdash \bot \\
x \Vdash \varphi \to \psi & \Leftrightarrow & \forall y\colon\ y \Vdash \varphi \Rightarrow x(y) \Vdash \psi \\
x \Vdash \forall x\varphi & \Leftrightarrow & \forall a \in D\colon\ x(a) \Vdash \varphi(\bar{a}) \\
x \Vdash \exists x\varphi & \Leftrightarrow & \exists a \in D\colon\ x \Vdash \varphi(\bar{a})
\end{array}
$$

   But now it's no longer obvious that $\vDash \varphi \vee \neg\varphi$ or $\neg\neg\varphi \vDash \varphi$.

2. **Meaning**: It's commonly assumed (following Wittgenstein) that the meaning of a sentence can be equated with its truth conditions. However, some (e.g., Dummett) think that grasping the meaning of a sentence crucially involves knowing when to assert/deny it. And one can be said to have obtained this ability if they understand the conditions under which the sentence in question is verified or falsified. Thus, one might think the meaning of a sentence is given by its verification conditions, not its truth conditions.

   On this view, to verify a disjunction, we must verify each disjunct, and to verify a conditional, we must show how any verification of the antecedent can be converted into a verification of the consequent. But then once again, we are no longer guaranteed LEM or double negation.

§4 DERIVATION RULES

The rules for intuitionistic logic are exactly the rules for classical logic *except* we drop the RAA rule:

$$\dfrac{\begin{matrix}[\neg\varphi]\\ \vdots\\ \bot\end{matrix}}{\varphi}\ \mathrm{RAA}$$

This means that we won't be able to derive $\varphi \vee \neg\varphi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi\vee\neg\varphi)]^2 \qquad \dfrac{[\varphi]^1}{\varphi\vee\neg\varphi}\,\vee\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\varphi}\,{\to}\mathrm{I}_1}{\varphi\vee\neg\varphi}\,\vee\mathrm{I} \qquad [\neg(\varphi\vee\neg\varphi)]^2}{\bot}\,{\to}\mathrm{E}}{\varphi\vee\neg\varphi}\ \mathrm{RAA}_2$$

Likewise, we cannot prove $\neg\neg\varphi \to \varphi$. To see this, note first that if we could, then we could prove LEM:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi\vee\neg\varphi)]^2 \qquad \dfrac{[\varphi]^1}{\varphi\vee\neg\varphi}\,\vee\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\varphi}\,{\to}\mathrm{I}_1}{\varphi\vee\neg\varphi}\,\vee\mathrm{I} \qquad [\neg(\varphi\vee\neg\varphi)]^2}{\bot}\,{\to}\mathrm{E}}{\neg\neg(\varphi\vee\neg\varphi)}\,{\to}\mathrm{I}_2}{\varphi\vee\neg\varphi}\ \neg\neg\mathrm{E}$$

Second, the direct proofs fails:

$$\dfrac{\dfrac{\dfrac{[\neg\neg\varphi]^1 \qquad [\neg\varphi]^2}{\bot}\,{\to}\mathrm{E}}{\varphi}\ \mathrm{RAA}_2}{\neg\neg\varphi\to\varphi}\,{\to}\mathrm{I}_1$$

$$\dfrac{\dfrac{\varphi\vee\neg\varphi \qquad [\varphi]^1 \qquad \dfrac{\dfrac{[\neg\varphi]^2 \qquad [\neg\neg\varphi]^3}{\bot}\,{\to}\mathrm{E}}{\varphi}\,\bot}{\varphi}\,\vee\mathrm{E}_{1,2}}{\neg\neg\varphi\to\varphi}\,{\to}\mathrm{I}_3$$

Here are some facts about what you cannot prove in intuitionistic logic:

- $\nvdash_i \neg(\varphi\wedge\psi)\to\neg\varphi\vee\neg\psi$
- $\nvdash_i \neg\forall x\varphi\to\exists x\,\neg\varphi$
- $\nvdash_i \forall x\,\neg\neg\varphi\to\neg\neg\forall x\varphi$
- $\nvdash_i (\neg\psi\to\neg\varphi)\to(\varphi\to\psi)$
- $\nvdash_i (\varphi\to\psi)\to(\neg\varphi\vee\psi)$
- $\nvdash_i \neg(\varphi\to\psi)\to(\varphi\wedge\neg\psi)$
- $\nvdash_i ((\varphi\to\psi)\to\varphi)\to\varphi$; thus, $\nvdash_i (\neg\varphi\to\varphi)\to\varphi$ Peirce's Law
- $\nvdash_i (\neg\neg\varphi\to\varphi)\to(\varphi\vee\neg\varphi)$

However, we still get these:

- $\vdash_i \neg(\varphi\wedge\neg\varphi)$
- $\vdash_i \neg\neg(\varphi\vee\neg\varphi)$
- $\vdash_i \varphi\to\neg\neg\varphi$
- $\vdash_i \neg\neg\neg\varphi\to\neg\varphi$
- $\vdash_i \neg\varphi\leftrightarrow(\varphi\to\bot)$
- $\vdash_i (\neg\varphi\vee\neg\psi)\to\neg(\varphi\wedge\psi)$
- $\vdash_i \neg(\varphi\vee\psi)\leftrightarrow(\neg\varphi\wedge\neg\psi)$
- $\vdash_i \neg(\neg\varphi\wedge\neg\psi)\leftrightarrow\neg\neg(\varphi\vee\psi)$
- $\vdash_i \neg\exists x\varphi\leftrightarrow\forall x\,\neg\varphi$
- $\vdash_i \exists x\,\neg\varphi\to\neg\forall x\varphi$
- $\vdash_i \neg\neg\forall x\varphi\to\forall x\,\neg\neg\varphi$
- $\vdash_i (\neg\varphi\vee\psi)\to(\varphi\to\psi)$
- $\vdash_i (\varphi\wedge\neg\psi)\to\neg(\varphi\to\psi)$
- $\vdash_i (\varphi\to\psi\wedge\psi\to\theta)\to(\varphi\to\theta)$
- $\vdash_i (\varphi\to\psi)\to(\neg\varphi\to\neg\psi)$
- $\vdash_i \neg\neg(\varphi\to\psi)\leftrightarrow(\neg\neg\varphi\to\neg\neg\psi)$
- $\vdash_i \neg\neg(\varphi\wedge\psi)\leftrightarrow(\neg\neg\varphi\wedge\neg\neg\psi)$

§5 THEOREMS ABOUT DERIVABILITY

Theorem *(Conservativity)*. If $\Gamma \vdash_i \varphi$, then $\Gamma \vdash_c \varphi$.

Theorem *(Substitution for Derivability)*. If $\Gamma \vdash_i \sigma$, then where p is a proposition variable and φ is a formula, $\Gamma[\varphi/p] \vdash_i \sigma[\varphi/p]$ (as long as the free variables in φ do not occur bound in these substitutions).

Proof: By induction on the length of the derivation. ■

Theorem *(Substitution for Equivalence)*. If $\Gamma, \varphi \leftrightarrow \psi \vdash_i \theta[\varphi/p] \leftrightarrow \theta[\psi/p]$ (as long as the free variables in φ and ψ do not occur bound in these substitutions).

Definition *(Negative Formula)*. A formula φ is ***negative*** if it does not contain $\vee$ or $\exists$, and all the atoms (except $\bot$) are in the immediate scope of a negation.

Theorem *(Conditions for $\neg\neg E$)*. If φ is negative, then $\neg\neg\varphi \vdash_i \varphi$.

Proof: By induction. Atomic case is taken care of by the fact that $\vdash_i \neg\neg\neg\varphi \to \neg\varphi$. As for $\bot$:

$$\dfrac{\dfrac{[\neg\neg\bot]^1 \qquad \dfrac{[\bot]^2}{\neg\bot}\,{\to}\mathrm{I}_2}{\bot}\,{\to}\mathrm{E}}{\neg\neg\bot \to \bot}\,{\to}\mathrm{I}_1$$

The inductive cases are taken care of using Lemma 6.2.2. ■

§6 FROM CLASSICAL TO INTUITIONISTIC LOGIC

Definition *(Gödel Translation)*. The ***Gödel translation*** of formulas $^\circ : \mathit{FORM} \to \mathit{FORM}$ is a map from classical formulas to intuitionistic formulas defined as follows:

$$\begin{array}{rcl}
\bot^\circ & = & \bot \\
\varphi^\circ & = & \neg\neg\varphi \text{ for atomic } \varphi \\
(\neg\varphi)^\circ & = & \neg\varphi^\circ \\
(\varphi \wedge \psi)^\circ & = & \varphi^\circ \wedge \psi^\circ \\
(\varphi \vee \psi)^\circ & = & \neg(\neg\varphi^\circ \wedge \neg\psi^\circ) \\
(\varphi \to \psi)^\circ & = & \varphi^\circ \to \psi^\circ \\
(\forall x\varphi)^\circ & = & \forall x\varphi^\circ \\
(\exists x\varphi)^\circ & = & \neg\forall x\,\neg\varphi^\circ.
\end{array}$$

Lemma *(Gödel Translation is Negative)*. φ° is negative. Hence, $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi^\circ$.

Lemma *(Classical Equivalence)*. $\vdash_c \varphi \leftrightarrow \varphi^\circ$.

Proof: Proof by induction. ■

Lemma *(Intuitionistic Equivalence)*. $\vdash_i \varphi \leftrightarrow \varphi^\circ$ if φ is negative.

Proof: If φ is negative, then φ° only differs by add two negations to atomics. But since all atomics are in the immediate scope of a negation, by substitution, we can drop those two negations, since $\vdash_i \neg\neg\neg\varphi \leftrightarrow \neg\varphi$. ■

Theorem *(Gödel Translation is Accurate)*. $\Gamma \vdash_c \varphi$ iff $\Gamma^\circ \vdash_i \varphi^\circ$.

Proof: $\Leftarrow$ follows from conservativity and classical equivalence. For $\Rightarrow$, we do proof by induction on the length of the derivation. The base case is trivial. Here are two samples for the inductive cases.

Suppose we're given the last rule is $\to$E:

$$\dfrac{\begin{matrix}\Gamma\\ \mathcal{D}_1\\ \varphi\end{matrix} \qquad \begin{matrix}\Gamma\\ \mathcal{D}_2\\ \varphi \to \psi\end{matrix}}{\psi} \to\text{E}$$

Then the following is valid by induction (and since $(\varphi \to \psi)^\circ = \varphi^\circ \to \psi^\circ$):

$$\dfrac{\begin{matrix}\Gamma^\circ\\ \mathcal{D}_1^\circ\\ \varphi^\circ\end{matrix} \qquad \begin{matrix}\Gamma^\circ\\ \mathcal{D}_2^\circ\\ (\varphi \to \psi)^\circ\end{matrix}}{\psi^\circ} \to\text{E}$$

Next, suppose the last rule is RAA:

$$\dfrac{\begin{matrix}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{matrix}}{\varphi}\ \text{RAA}$$

Then we have that:

$$\dfrac{\begin{matrix}[\neg\varphi^\circ]\\ \mathcal{D}^\circ\\ \bot\end{matrix}}{\neg\neg\varphi^\circ} \to\text{I}$$

But since φ° is negative, $\neg\neg\varphi^\circ \vdash_i \varphi$. ■

Theorem *(Intuitionistic Conservativity over Negative Fragment)*. If φ is negative, then $\vdash_c \varphi$ iff $\vdash_i \varphi$.

Proof: $\Rightarrow$ follows from conservativity. As for $\Leftarrow$, since φ is negative, $\vdash_i \varphi \leftrightarrow \varphi^\circ$. Hence, $\vdash_i \varphi^\circ$. By the accuracy of the Gödel translation, $\vdash_c \varphi^\circ$. But by classical equivalence, $\vdash_c \varphi \leftrightarrow \varphi^\circ$. Hence, $\vdash_c \varphi$. ■

Theorem *(Glivenko's Theorem)*. For all propositional φ, $\vdash_c \varphi$ iff $\vdash_i \neg\neg\varphi$.

Proof: $\Leftarrow$ is easy. For $\Rightarrow$, it suffices to show that $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi$, since then we can apply the accuracy of the Gödel translation to get $\vdash_c \varphi$. The atomic case is trivial.

Negation. If $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi$, then $\vdash_i \neg\varphi^\circ \leftrightarrow \neg\neg\neg\varphi$. But $\neg\varphi^\circ = (\neg\varphi)^\circ$. ✓

Conjunction. If $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi$ and $\vdash_i \psi^\circ \leftrightarrow \neg\neg\psi$, then $\vdash_i (\varphi \wedge \psi)^\circ \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$. But $\vdash_i \neg\neg(\varphi \wedge \psi) \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$. ✓

Disjunction. If $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi$ and $\vdash_i \psi^\circ \leftrightarrow \neg\neg\psi$, then $\vdash_i (\varphi \vee \psi)^\circ \leftrightarrow \neg(\neg\neg\neg\varphi \wedge \neg\neg\neg\psi) \leftrightarrow \neg\neg\neg(\neg\varphi \wedge \neg\psi)$. But since $\vdash_i \neg\varphi \wedge \neg\psi \leftrightarrow \neg(\varphi \vee \psi)$, it follows that $\vdash_i (\varphi \vee \psi)^\circ \leftrightarrow \neg\neg\neg\neg(\varphi \vee \psi) \leftrightarrow \neg\neg(\varphi \vee \psi)$. ✓

Conditional. $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi$ and $\vdash_i \psi^\circ \leftrightarrow \neg\neg\psi$, then $\vdash_i (\varphi \to \psi)^\circ \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$. But $\vdash_i \neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$. ✓ ■

This result does not extend to predicate logic. If we tried to extend the induction to the quantifier cases, the $\exists$ would go through, but we'd get stuck on the $\forall$ case, since $\nvdash_i \forall x\, \neg\neg\varphi \to \neg\neg\forall x\, \varphi$.