Compactness and Löwenheim-Skolem

Phil 140A Spring 2016

1. Suppose a sentence σ is true in every infinite model. Show that there's an $n \in \mathbb{N}$ such that σ is true in every model of at least size n.

> **Proof**: Suppose there's no $n \in \mathbb{N}$ such that σ is true in every model of at least size n. Then for each $n \in \mathbb{N}$, there is a $\mathfrak{A}_n$ of at least size n such that $\mathfrak{A}_n \not\models \sigma$, i.e., $\mathfrak{A}_n \models \neg\sigma$. This means that $\neg\sigma$ has arbitrarily large finite models. But then by Lemma 4.2.6, it follows that $\neg\sigma$ has an infinite model. So σ is not true in every infinite model. ■

2. Let $\varphi(x)$ be a formula. For any model $\mathfrak{A}$, define $\varphi(\mathfrak{A}) := \{a \in A \mid \mathfrak{A} \models \varphi(\bar{a})\}$ (the set of elements in $\mathfrak{A}$ satisfying φ). Suppose for each $n \in \mathbb{N}$, there is a model $\mathfrak{A}_n$ such that $|\varphi(\mathfrak{A}_n)| \geqslant n$. Show that there's a model $\mathfrak{A}$ such that $|\varphi(\mathfrak{A})| \geqslant \aleph_0$ (i.e., $\varphi(\mathfrak{A})$ is infinite).

> **Proof**: Consider the set of sentences $\Gamma = \{\exists_{\geqslant n} x\, \varphi(x) \mid n \in \mathbb{N}\}$ (that is, the set of sentences that says "there are at least n-many φ's"). First note that Γ is finitely satisfiable. For suppose $\Gamma_0 \subseteq \Gamma$ is finite. Then there's a max n such that $\exists_{\geqslant n} x\, \varphi(x) \in \Gamma_0$. But then $\mathfrak{A}_n \models \Gamma_0$. So by compactness, Γ has a model, and that model must contain an infinite number of φ's. ■

3. Suppose there's a model $\mathfrak{A}$ such that $\varphi(\mathfrak{A})$ is infinite. Show that for any infinite cardinal number λ, there's a model $\mathfrak{B}$ such that $\mathfrak{B}$ satisfies the same sentences as $\mathfrak{A}$ and $\varphi(\mathfrak{B})$ is exactly of size λ.

> **Proof**: Suppose $\varphi(\mathfrak{A})$ is of size κ, where κ is infinite. Let $\{c_\alpha \mid \alpha < \lambda\}$ be a set of new constants. Consider the following set of sentences:
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> $$\Gamma := Th(\mathfrak{A}) \cup \{c_\alpha \neq c_\beta \mid \alpha \neq \beta \text{ and } \alpha, \beta < \lambda\} \cup \{\varphi(c_\alpha) \mid \alpha < \lambda\}.$$
>
> Γ is finitely satisfiable: in fact, for any finite subset $\Gamma_0 \subseteq \Gamma$, we can let $\mathfrak{A}$ be our model, where we just assign each c_α occurring in Γ_0 to a distinct φ. So by compactness, Γ is satisfiable, and hence, there is a model $\mathfrak{B}$ of Γ. Now, $\mathfrak{B}$ *may have more than λ-many φ's*. But now by Downward Löwenheim-Skolem, we can find a model of size λ that make all the same sentences true as $\mathfrak{B}$. So there's a model of size λ of Γ. And since there's at least λ-many φ's in any model of Γ, it follows that such a model has exactly λ-many φs. ■

4. Let P and Q be unary-predicates. Show that there is no first-order sentence μ that is true in a model $\mathfrak{A}$ iff $|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})| < |P(\mathfrak{A}) \cap Q(\mathfrak{A})|$ (i.e., most Ps are Qs in $\mathfrak{A}$).

Proof: Suppose for *reductio* there is such a μ. Consider the set of sentences:

$$\Gamma := \{\mu\} \cup \{\exists_{\geqslant n} x\,(P(x) \wedge Q(x)) \mid n \in \mathbb{N}\} \cup \{\exists_{\geqslant n} x\,(P(x) \wedge \neg Q(x)) \mid n \in \mathbb{N}\}\,.$$

Γ is finitely satisfiable. To see this, let $\Gamma_0 \subseteq \Gamma$ be finite. Then there's a max n such that $\exists_{\geqslant n} x\,(P(x) \wedge \neg Q(x)) \in \Gamma_0$ and a max m such that $\exists_{\geqslant m} x\,(P(x) \wedge Q(x)) \in \Gamma_0$. Let $k = \max(n, m)$. Define a model:

$$\mathfrak{A}_k = (\{1, \ldots, 2k+1\}, \{1, \ldots, 2k+1\}, \{1, \ldots, k+1\})\,.$$

So everything is P and only $1, \ldots, k+1$ are Q. Then:

$$|P(\mathfrak{A}) \cap Q(\mathfrak{A})| = |Q(\mathfrak{A})| = k+1.$$

But:

$$|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})| = |\neg Q(\mathfrak{A})| = (2k+1) - (k+1) = k.$$

So $|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})| < |P(\mathfrak{A}) \cap Q(\mathfrak{A})|$. And since $k = \max(n, m)$, and since both sets above are of at least size k, it follows that $\mathfrak{A}_k \models \mu \wedge \exists_{\geqslant n} x\,(P(x) \wedge \neg Q(x)) \wedge \exists_{\geqslant m} x\,(P(x) \wedge Q(x))$. So $\mathfrak{A}_k \models \Gamma_0$.

So by compactness, Γ is satisfiable. So there's a model $\mathfrak{A} \models \Gamma$. Now, in any model of Γ, $|P(\mathfrak{A}) \cap Q(\mathfrak{A})| \geqslant \aleph_0$ and $|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})| \geqslant \aleph_0$. So $\mathfrak{A}$ has to be infinite. But we cannot yet derive a contradiction from this, since it might be that $|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})| < |P(\mathfrak{A}) \cap Q(\mathfrak{A})|$ (in particular, $|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})|$ might be a smaller infinite cardinal than $|P(\mathfrak{A}) \cap Q(\mathfrak{A})|$).

But we can derive a contradiction using Downward Löwenheim-Skolem. For by Downward Löwenheim-Skolem, there is a *countable* model $\mathfrak{B}$ that makes all the same first-order sentences true as $\mathfrak{A}$. Hence, $\mathfrak{B} \models \Gamma$, and in particular, $\mathfrak{B} \models \mu$. But if $\mathfrak{B}$ is countable, and if $|P(\mathfrak{B}) \cap \neg Q(\mathfrak{B})| \geqslant \aleph_0$ and $|P(\mathfrak{B}) \cap Q(\mathfrak{B})| \geqslant \aleph_0$, then $|P(\mathfrak{B}) \cap \neg Q(\mathfrak{B})| = \aleph_0 = |P(\mathfrak{B}) \cap Q(\mathfrak{B})|$. So even though $\mathfrak{B} \models \mu$, we have that $|P(\mathfrak{B}) \cap \neg Q(\mathfrak{B})| \nless |P(\mathfrak{B}) \cap Q(\mathfrak{B})|$. ■