SET THEORY

PHIL 140A   SPRING 2016

DEFINITIONS

Name	Notation	Definition		
Axiom of Extensionality		$x = y$	$\leftrightarrow$	$\forall a\ (a \in x \leftrightarrow a \in y)$
Restricted Quantifiers	$\forall a \in x$	$\forall a \in x\ \varphi$	$=$	$\forall a\ (a \in x \rightarrow \varphi)$
	$\exists a \in x$	$\exists a \in x\ \varphi$	$=$	$\exists a\ (a \in x \wedge \varphi)$
Subset	$\subseteq$	$x \subseteq y$	$\leftrightarrow$	$\forall a\ (a \in x \rightarrow a \in y)$
Proper Subset	$\subset$	$x \subset y$	$\leftrightarrow$	$x \subseteq y \wedge x \neq y$
Intersection	$\cap$	$a \in x \cap y$	$\leftrightarrow$	$(a \in x \wedge a \in y)$
		$x \cap y$	$=$	$\{a \mid a \in x \wedge a \in y\}$
Union	$\cup$	$a \in x \cup y$	$\leftrightarrow$	$(a \in x \vee a \in y)$
		$x \cup y$	$=$	$\{a \mid a \in x \vee a \in y\}$
Complement	$-$	$a \in x - y$	$\leftrightarrow$	$(a \in x \wedge a \notin y)$
		$x - y$	$=$	$\{a \mid a \in x \wedge a \notin y\}$
Power set	$\mathcal{P}$	$a \in \mathcal{P}(x)$	$\leftrightarrow$	$a \subseteq x$
		$\mathcal{P}(x)$	$=$	$\{a \mid a \subseteq x\}$
Ordered Pair	(a,b)	(a,b)	$=$	$\{\{a\}, \{a,b\}\}$
Ordered n-Tuple		$(a_1, \ldots, a_n)$	$=$	$((a_1, \ldots, a_{n-1}), a_n)$
Big Union	$\bigcup$	$a \in \bigcup x$	$\leftrightarrow$	$\exists y\ (a \in y \wedge y \in x)$
		$\bigcup x$	$=$	$\{a \mid \exists y\ (a \in y \wedge y \in x)\}$
		$\bigcup_{i \in I} x_i$	$=$	$\bigcup \{x_i \mid i \in I\}$
Big Intersection	$\bigcap$	$a \in \bigcap x$	$\leftrightarrow$	$\forall y\ (y \in x \rightarrow a \in y)$
		$\bigcap x$	$=$	$\{a \mid \forall y\ (y \in x \rightarrow a \in y)\}$
		$\bigcap_{i \in I} x_i$	$=$	$\bigcap \{x_i \mid i \in I\}$
Cartesian Product	$\times$	$x \times y$	$=$	$\{(a,b) \mid a \in x \wedge b \in y\}$
Domain	dom	$\mathrm{dom}(R)$	$=$	$\{a \mid \exists b\ ((a,b) \in R)\}$
Range	ran	$\mathrm{ran}(R)$	$=$	$\{b \mid \exists a\ ((a,b) \in R)\}$
Composition	$\circ$	$(g \circ f)(a)$	$=$	$g(f(a))$
Equivalence Class	$[a]_R$	$[a]_R$	$=$	$\{b \mid R(a,b)\}$

PROPERTIES OF RELATIONS

Name	Definition
Reflexive	$\forall a\, R(a,a)$
Irreflexive	$\forall a\, \neg R(a,a)$
Symmetric	$\forall a\, \forall b\ (R(a,b) \rightarrow R(b,a))$
Asymmetric	$\forall a\, \forall b\ (R(a,b) \rightarrow \neg R(b,a))$
Anti-symmetric	$\forall a\, \forall b\ ((R(a,b) \wedge R(b,a)) \rightarrow a = b)$
Transitive	$\forall a\, \forall b\, \forall c\ ((R(a,b) \wedge R(b,c)) \rightarrow R(a,c))$
Euclidean	$\forall a\, \forall b\, \forall c\ ((R(a,b) \wedge R(a,c)) \rightarrow R(b,c))$
Connected	$\forall a\, \forall b\ (a \neq b \rightarrow (R(a,b) \vee R(b,a)))$

- A relation is an ***equivalence relation*** iff it's reflexive, symmetric, and transitive.

- (x, R) is a ***partial order*** iff $R \subseteq x \times x$ and is reflexive, anti-symmetric, and transitive.

- A partial order $(x, \leqslant)$ has a is a ***minimal element*** in y if there's no $b \in y$ where $b < a$.

- $(x, \leqslant)$ is ***well-founded*** if every nonempty $y \subseteq x$ has a minimal element.

EXAMPLES

Note: these proofs are purposely wordy so that my reasoning is clear. In the first couple of problem sets, it's good idea to go step-by-step and explain your reasoning clearly than to skip a bunch of steps. However, for the problem set, you don't necessarily need this much wordiness; as long as your reasoning is clearly stated, that's okay.

Exercise. Prove that $x \cup y = y \cup x$.

Exercise. Prove $x \cup y = y \cup x$ *without* the Axiom of Extensionality.
    Hint: use the alternative definition of union from the reading:

$$z = x \cup y \leftrightarrow \forall a\ (a \in z \leftrightarrow (a \in x \vee a \in y)).$$

Exercise. A relation is an ***equivalence relation*** if it's reflexive, symmetric, and transitive. Prove that a relation R is an equivalence relation iff it's reflexive and euclidean