Past Prelim Problems

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Abstract

What follows is a collection of (attempted) solutions to some of the past prelim problems in preparation for Part I (the Metamathematics part) of the prelims in the Group in Logic at UC Berkeley. The scope of these prelim problems includes model theory, recursion theory, and incompleteness results. Although most of the material is covered in the Math 225 series at Berkeley, some of the material goes beyond what is taught in those courses. For references, see the following recommendations:

(i) Model Theory: Hodges [2], Marker [4]

(ii) Recursion Theory: Rogers [5], Soare [6]

(iii) Arithmetic: Kaye [3] (and for fun, Boolos [1])

Text in green mark hyperlinks to other problems in the document, usually followed by a page number in case you're reading this on paper. The header is always hyperlinked to the beginning of the section. I've included the problem statements, for completeness, but have taken the liberty of rewording things here and there.

I cannot guarantee that these solutions are 100% accurate, and they are currently incomplete. The date on this page is the date of the most recent update. If you find errors, if you have solutions to problems that don't have on here, or if you have an alternative solution, please let me know![1]

[1] Many thanks to Russell Buehler, Alex Kruckman, Adam Leśnikowski, Lisha Li, and Michael Wan for their suggestions, corrections, and insight in working through these problems. Also, many thanks to Tom Scanlon and John Steel for their guidance and hints.

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Notes on Notation

Model Theory: If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$, etc. are models, then their respective domains are A, B, M, etc.. The diagram of $\mathcal{A}$ is denoted "$\mathsf{Diag}(\mathcal{A})$", the elementary diagram "$\mathsf{ElDiag}(\mathcal{A})$".

When building Ehrenfeucht-Mostowski models, I follow Hodges [2] and call the linear order used to build the Ehrenfeucht-Mostowski model the **spine** of the model. If $\langle I, < \rangle$ is the spine of the model, I use "$\mathrm{Hull}(I)$" to denote the Skolem hull over the order-indiscernibles indexed by I.

Recursion Theory: As usual, "ϕ_e" denotes the e^{th} recursive function in some fixed enumeration. "$\phi_e(k) \downarrow$" is true when ϕ_e on input k eventually converges; "$\phi_e(k) \uparrow$" is true when ϕ_e on input k diverges. "$\phi_{e,s}(k) \downarrow$" is true when ϕ_e on input k converges by stage s of the computation. "W_e" denotes the domain of ϕ_e, i.e. the e^{th} r.e set in some fixed enumeration, and "$W_{e,s}$" denotes the s^{th} stage of the construction of W_e.

If f is a function, then "ϕ_e^f" denotes the e^{th} f-recursive function, i.e. the e^{th} function that's recursive relative to oracle f. If A is a set, then "ϕ_e^A" denotes the e^{th} A-recursive function. Similarly for "W_e^f" and "W_e^A".

In the context of recursion theory, if A is a set, then "$A(n)$" means "$n \in A$", and "$\neg A(n)$" means "$n \notin A$". The characteristic function of A, χ_A, is defined as follows:

$$\chi_A(n) = \begin{cases} 1 & \text{if } A(n) \\ 0 & \text{if } \neg A(n) \end{cases}$$

We let "$\overline{A}$" denote the complement of A (relative to $\mathbb{N}$), so "$\overline{A}(n)$" is the same as "$\neg A(n)$".

The following is a list of sets (and their complexity) that appear:

(Σ^0_1) $K := \{e \mid \phi_e(e) \downarrow\} = \{e \mid W_e(e)\}$

(Σ^0_2) $\mathsf{Fin} := \{e \mid |W_e| < \aleph_0\}$

(Π^0_2) $\mathsf{Inf} := \{e \mid |W_e| = \aleph_0\}$

(Π^0_2) $\mathsf{Tot} := \{e \mid \forall x\, \phi_e(x) \downarrow\} = \{e \mid W_e = \mathbb{N}\}$

(Σ^0_3) $\mathsf{Cof} := \{e \mid W_e \text{ is cofinite}\} = \{e \mid |\mathbb{N} - W_e| < \aleph_0\}$

(Σ^0_3) $\mathsf{Rec} := \{e \mid W_e \text{ is recursive}\}$

Peano Arithmetic: PA is the theory of Peano arithmetic, Q is the theory of Robinson arithmetic (i.e. PA without an induction schema). If $n \in \mathbb{N}$, then "$\underline{n}$" denotes the numeral corresponding to n in the language of PA. If φ is a formula, then $\ulcorner \varphi \urcorner$ is the gödel number of φ in some standard fixed coding. For readability, I also use $\ulcorner \varphi \urcorner$ for the gödel numeral (as opposed to $\underline{\ulcorner \varphi \urcorner}$).

The following is a list of functions, formulae, and sentences (and their complexity) that appear throughout:

(Δ_1^0) $\mathsf{Seq}\,(s) :=$ "s codes a sequence of numbers"

(Δ_1^0) $\mathsf{lh}\,(s) :=$ "the length of sequence s"

(Δ_1^0) $\mathsf{Prf}_T\,(s, x) :=$ "s codes a proof in T of sentence x"

(Σ_1^0) $\mathsf{Prv}_T\,(x) := \exists s\, \mathsf{Prf}_T\,(s, x)$

(Π_1^0) $\mathsf{Con}\,(T) := \neg \mathsf{Prv}_T\,(\ulcorner \bot \urcorner)$

(Σ_n^0) $\mathsf{Sat}_{\Sigma_n^0}\,(x, y) :=$ "x codes a Σ_n^0 sentence that's satisfied by y"

Throughout, I simply assume PA is both consistent and pretty smart, without proof.

June 2013

WARNING: Since I took the June 2013 prelim, I was required to write solutions to these problems. And because I'm OCD, some of these answers are extremely lengthy, esp. problems 2 and 3 (though, to be honest, I didn't see any other way...). Earlier prelims will be more representative.

1. (a) Show that there are two disjoint Σ^0_2 sets A and B of natural numbers such that there is no Δ^0_2 set C with $A \subseteq C$ and $C \cap B = \varnothing$.

   (b) Show that there is no complete Σ^0_1-sound $T \supseteq \mathsf{PA}$ such that T is Δ^0_2.

▶ **ANSWER (a):** Let S be a Σ^0_1 set. Define the sets:

$$\begin{aligned} A &\coloneqq \{e \mid \phi^S_e(e) = 0\} \\ B &\coloneqq \{e \mid \phi^S_e(e) = 1\} \end{aligned}$$

Suppose $C \supseteq A$ was Δ^0_2 and $C \cap B = \varnothing$. Then for some d:

$$\phi^S_d(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

But now consider whether $d \in C$:

$$\begin{aligned} d \in C &\Rightarrow \phi^S_d(d) = 1 \Rightarrow d \in B \Rightarrow C \cap B \neq \varnothing \Rightarrow \bot \\ d \notin C &\Rightarrow \phi^S_d(d) = 0 \Rightarrow d \in A \Rightarrow A \nsubseteq C \qquad \Rightarrow \bot \end{aligned}$$

Hence, there cannot be such a Δ^0_2 set C.[2] □

[2] This is basically the proof that are two Σ^0_1 recursively-inseparable sets, except with oracles. The proof generalizes to any Σ^0_n for $n \in \omega$, *mutatis mutandis*.

▶ **ANSWER (b):** First, let's recall some definitions. If $T \supseteq \mathsf{PA}$ and $A \subseteq \mathbb{N}$, we say that T **weakly represents** A with formula $\alpha(x)$ if $A(n) \Leftrightarrow T \vdash \alpha(\underline{n})$. We say T **strongly represents** A with $\alpha(x)$ if T weakly represents A with $\alpha(x)$ and in addition $\neg A(n) \Leftrightarrow T \vdash \neg\alpha(\underline{n})$. It's a standard result that PA weakly represents all Σ^0_1 sets, and strongly represents all Δ^0_1 sets.

Now, let $T \supseteq \mathsf{PA}$ be complete and Σ^0_1-sound.

CLAIM (1): T strongly represents every Σ^0_1 set with some Σ^0_1-formula.

▶ **PROOF (1):** Suppose A is a Σ^0_1 set. Since PA weakly represents A, there is a Σ^0_1-formula $\alpha(x)$ such that, for all $n \in \mathbb{N}$, $A(n) \Leftrightarrow \mathsf{PA} \vdash \alpha(\underline{n})$. Since $\mathsf{PA} \subseteq T$, $A(n) \Rightarrow T \vdash \alpha(\underline{n})$. And since T is Σ^0_1-sound, $T \vdash \alpha(\underline{n}) \Rightarrow A(n)$. So T weakly represents A with $\alpha(x)$.

We need to also show: $\neg A(n) \Leftrightarrow T \vdash \neg\alpha(\underline{n})$

($\Leftarrow$) Suppose $T \vdash \neg\alpha(\underline{n})$. Since T is consistent, T is Π^0_1-sound, and hence $\neg A(n)$. □

($\Rightarrow$) Suppose $\neg A(n)$. Since T is complete, either $T \vdash \alpha(\underline{n})$ or $T \vdash \neg\alpha(\underline{n})$. If the former, then by Σ^0_1-soundness, $A(n)$ would be true, $\perp$. Hence we must have $T \vdash \neg\alpha(\underline{n})$. □

It's straightforward to check that this implies that T also strongly represents every Π^0_1 set with a Π^0_1-formula.

Now, let A and B be Σ^0_2 inseparable sets (which exist by part (a)). Let's say that:

$$A(n) \Leftrightarrow \exists x\, A'(x,n)$$
$$B(n) \Leftrightarrow \exists x\, B'(x,n)$$

where A' and B' are Π^0_1 relations. Let T represent A' and B' with the Π^0_1-formulae α and β respectively. Now, define the set:

$$C := \{n \mid T \vdash \exists x\ (\alpha(x,\underline{n}) \wedge \forall z < x\ \neg\beta(z,\underline{n}))\}$$

For brevity, let $\gamma(u) := \exists x\ (\alpha(x,u) \wedge \forall z < x\ \neg\beta(z,u))$, so that we have $C = \{n \mid T \vdash \gamma(\underline{n})\}$. Notice that γ is Σ^0_2.

Suppose for *reductio* that T was Δ^0_2. We'll show that C Δ^0_2-separates A and B, contrary to hypothesis.

C is Δ^0_2: Since T was Δ^0_2—that is, for any sentence φ, we can decide whether or not $T \vdash \varphi$ in a Δ^0_2 way—C will be as well. ✓

$A \subseteq C$: If $n \in A$, then $n \notin B$, since $A \cap B = \varnothing$. So $\forall z\, \neg\beta(z, n)$, and hence $\gamma(n)$, would be true.

CLAIM (2): T weakly represents every Σ^0_2 set with some Σ^0_2-formula.

▶ **PROOF (2):** Let S be a Σ^0_2 set, where $S(n) \Leftrightarrow \exists x\, R(x, n)$ for some Π^0_1 set R. Let T represent R with the Π^0_1-formula $\rho(x, y)$. We'll show $S(n) \Leftrightarrow T \vdash \exists x\, \rho(x, \underline{n})$, which will suffice.

($\Rightarrow$) Suppose $S(n)$, i.e. $\exists x\, R(x, n)$, is true. Then there is a $k \in \mathbb{N}$ such that $R(k, n)$. Since T strongly represents R, $T \vdash \rho(\underline{k}, \underline{n})$. Hence, $T \vdash \exists x\, \rho(x, \underline{n})$. □

($\Leftarrow$) Suppose $T \vdash \exists x\, \rho(x, \underline{n})$, and suppose for *reductio* that $\neg S(n)$. Then $\forall x\, \neg R(x, n)$ is true, i.e. for each $k \in \mathbb{N}$, $\neg R(k, n)$. Since T strongly represents R, $T \vdash \neg\rho(\underline{k}, \underline{n})$ for each $k \in \mathbb{N}$. But then T is ω-inconsistent, which can't be since T is Σ^0_1-sound, $\perp$.[a] □

[a] The ($\Leftarrow$) direction is unnecessary for this problem, but I've provided a proof anyway.

So by ($\Rightarrow$), $T \vdash \gamma(\underline{n})$, i.e. $n \in C$. ✓

$B \cap C = \varnothing$: Suppose $n \in B$. Then $\exists x\, B'(x, n)$ is true. So there is a least $k \in \mathbb{N}$ such that $B'(k, n)$ is true. Since this is Π^0_1, $T \vdash \beta(\underline{k}, \underline{n})$. And since $A \cap B = \varnothing$, $\forall z < k\, \neg\alpha(z, n)$. This is Σ^0_1, so $T \vdash \forall z < \underline{k}\, \neg\alpha(z, \underline{n})$. Hence, $T \vdash \exists x\ (\beta(x, \underline{n}) \wedge \forall z < x\, \neg\alpha(z, \underline{n}))$. This sentence entails $\neg\gamma(\underline{n})$, so $T \vdash \neg\gamma(\underline{n})$. Since T is consistent, $n \notin C$. ✓

Hence, C would separate A and B if T were Δ^0_2, $\perp$.[3] □

[3] This proof mimics the proof that Q is essentially undecidable.

2. Show that there is an A such that $\varnothing <_T A <_T K$.

Note: I present two different proofs. Both proceed by proving that there are two Turing-incomparable sets (that is, two sets A and B such that $A \not\leqslant_T B$ and $B \not\leqslant_T A$). Neither can be Turing-complete, so if A and B are r.e, then this suffices. The first proof is easier, but it alone doesn't guarantee these sets are r.e. The second one is harder, but does ensure they're r.e.

▶ **Answer (Kleene-Post):** We will proceed by constructing χ_A and χ_B in stages so that that $\chi_A = \bigcup_i f_i$ and $\chi_B = \bigcup_i g_i$. At the end of the construction, we want the following requirements to be met for all e:

Requirement R_{2e}: $\phi_e^{\chi_A} \neq \chi_B$

Requirement R_{2e+1}: $\phi_e^{\chi_B} \neq \chi_A$

Notice that our oracles are characteristic functions, *not* sets. This means that if $\phi_e^f(n)\downarrow$ by stage s, then for any $g \supseteq f$, $\phi_e^g(n)\downarrow = \phi_e^f(n)$ by stage s as well. This is crucial for Case 1 below.

Stage -1: Set $f_{-1} = g_{-1} = \varnothing$.

Stage $2e$: We will ensure that R_{2e} is satisfied. There are two cases to consider.

Case 1: There is a $k \notin \mathrm{dom}\,(g_{2e-1})$ and a $u \supseteq f_{2e-1}$ such that $\phi_e^u(k)\downarrow$. Then pick such a u and set $f_{2e} = u$. Now pick an $i \in \{0,1\}$ such that $\phi_e^{f_{2e}}(k) \neq i$, and set $g_{2e} = g_{2e-1} \cup \{\langle k, i\rangle\}$.

Case 2: Otherwise. Then just set $f_{2e} = f_{2e-1}$ and $g_{2e} = g_{2e-1}$.

Stage $2e+1$: We will ensure that R_{2e+1} is satisfied. There are two cases to consider.

Case 1: There is a $k \notin \mathrm{dom}\,(f_{2e})$ and a $u \supseteq g_{2e}$ such that $\phi_e^u(k)\downarrow$. Then pick such a u and set $g_{2e+1} = u$. Now pick an $i \in \{0,1\}$ such that $\phi_e^{g_{2e+1}}(k) \neq i$, and set $f_{2e+1} = f_{2e} \cup \{\langle k, i\rangle\}$.

Case 2: Otherwise. Then just set $f_{2e+1} = f_{2e}$ and $g_{2e+1} = g_{2e}$.

At the end of the construction, we're guaranteed that each R_e is satisfied. So set $\chi_A = \bigcup_i f_i$ and $\chi_B \bigcup_i g_i$. Suppose, for example, $A \leqslant_T B$. Then $\phi_d^{\chi_B} = \chi_A$ for some d. But R_{2d+1} ensured this isn't the case, $\perp$. Similarly for $B \leqslant_T A$. Hence, neither A nor B are Turing-reducible to the other. ☐

▶ **ANSWER (FRIEDBERG-MUCHNIK):** The following proof uses a priority argument (see Rogers [5, chp. 10.2]).

The Idea

We will construct two r.e sets A, B in stages so that $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$, where A_s is what we've constructed of set A by stage s, and similarly for B_s. Throughout the construction, we will ensure that each of the following requirements is satisfied eventually for each e:

Requirement R_{2e}**:** There is a number n_{2e} such that either $\phi_e^B(n_{2e}) \uparrow$ or we have that $\phi_e^B(n_{2e}) = 0 \Leftrightarrow n_{2e} \in A$.

Requirement R_{2e+1}**:** There is a number n_{2e+1} such that either $\phi_e^A(n_{2e+1}) \uparrow$ or we have that $\phi_e^A(n_{2e+1}) = 0 \Leftrightarrow n_{2e+1} \in B$.

The idea is that these requirements will guarantee that any function using B as an oracle will fail to be the characteristic function of A at some point (and *vice versa*). Suppose each R_d is satisfied by our construction, and suppose for *reductio* that $A \leqslant_T B$. That means that there is an e such that $\phi_e^B = \chi_A$. But then by R_{2e}, for some n_{2e}, we must have one of two cases:

(i) $\phi_e^B(n_{2e}) \uparrow$: Since χ_A is total, $\phi_e^B \neq \chi_A$, $\perp$.

(ii) $\phi_e^B(n_{2e}) = 0 \Leftrightarrow n_{2e} \in A$: Since $n_{2e} \in A \Leftrightarrow \chi_A(n_{2e}) = 1$, we'll have that $\phi_e^B(n_{2e}) = 0 \Leftrightarrow \chi_A(n_{2e}) = 1 \Leftrightarrow \chi_A(n_{2e}) \neq 0$, so again $\phi_e^B \neq \chi_A$, $\perp$.

Hence, there cannot be such an e, and thus $A \nleqslant_T B$. By similar reasoning, $B \nleqslant_T A$. Thus, it suffices to present a construction of two r.e (nonrecursive) sets A and B in which each R_i is eventually satisfied.

The Construction

To help us in this construction, we introduce a set of "movable markers" $m_0, m_1, m_2, \ldots$, which may be associated with numbers at any given stage, and can be reassociated from stage to stage. If m_d is a marker, we'll let "$m_{d,s}$" denote the number that m_d is associated with at stage s. The goal is to ensure that, at the end of the construction, m_d is associated with a number that witnesses R_d.

During the construction of A and B, we may find that, at some previous stage s, we ensured that $m_{d,s}$ is a witness to R_d, but at some later stage t, we need to move that marker m_d to another number $m_{d,t}$ in order to satisfy a different requirement R_c, thus losing our original witness to R_d. Thus, sometimes, the satisfaction of different requirements will come into conflict. To resolve this issue, we will give *priority* to the requirement with the smallest index. While this may mess up the satisfaction of requirements with higher indices, we'll show that this only happens *at most* finitely often. Thus, we'll ensure that each movable marker eventually comes to a permanent halt, which will guarantee that the requirement it's associated with is satisfied for the rest of the construction.

To help us keep track of which numbers are in A and B, we'll use *signs*, $+_A$, $+_B$, $-_A$, $-_B$. At any stage s, A_s will be the set of numbers with a $+_A$ sign by stage s (similarly for B_s and $+_B$). We'll then set $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$. Minuses are mostly for bookkeeping. A number with $-_A$ means it's *temporarily* not in A; but a $-_A$ may at some later point be changed to a $+_A$, if for instance that number is needed as a witness to a requirement of higher priority than the requirement that gave it the $-_A$ (similarly with $-_B$). **Note**: Signs are associated with *numbers*, not with their corresponding markers. **Remember**: m_d is a *marker*, while $m_{d,s}$ is a *number*.

Finally, for brevity, we'll introduce some definitions. A number is **unmarked** if no marker is associated with it; otherwise it's **marked**. A number is **S-unsigned** if it both lacks a $+_S$ and lacks a $-_S$; otherwise it is **S-signed**. Similarly, we'll use the terms **S-positive**, **S-negative**, **S-nonpositive**, etc. as one would expect. Finally, we'll say a number is **free in S** if it is unmarked and neither it *nor any number after it* is S-signed.

We now give the details of the construction. At each stage, we must (i) assign new markers to numbers, (ii) add signs in order to fulfill some new requirement, if possible, and (iii) reassign markers accordingly.

Stage 0: Set $A_0 = B_0 = \varnothing$. No markers are associated with any numbers yet.

Stage $s+1$: Suppose we've completed stage s. Thus, A_s and B_s are defined, and the markers $m_0, \ldots, m_{s-1}$ are exactly the markers assigned to numbers (i.e. $m_{0,s}, \ldots, m_{s-1,s}$ are defined, but not $m_{i,s}$ for $i \geqslant s$). There are two cases regarding what to do at stage $s+1$.

Case $s = 2e$**:** First, set $m_{s,s}$ to be the first number free in A after $m_{s-2,s}$. Next, search for a $d \leqslant e$ such that:

(i) $\phi_{d,e}^{B_s}(m_{2d,s}) \downarrow = 0$, and

(ii) $m_{2d,s}$ is A-nonpositive

If there is no such d, set $m_{i,s+1} = m_{i,s}$ for all $i \leqslant s$, and go to the next stage. Otherwise, pick the least such d, and give $m_{2d,s}$ the sign $+_A$. In addition, give the sign $-_B$ to each B-nonpositive number used in the computation of $\phi_{d,e}^{B_s}(m_{2d,s})$. Finally, reassociate markers as follows. For all $i \leqslant 2d$, set $m_{i,s+1} = m_{i,s}$ (i.e. don't move them). For all i such that $d \leqslant i < e$ (if there are any), set $m_{2i+1,s+1}$ to be the first $e-d$ numbers that are free in B (in increasing order). Don't move any of the even-indexed markers.

Case $s = 2e+1$**:** First, set $m_{s,s}$ to be the first number free in B after $m_{s-2,s}$. Next, search for a $d \leqslant e$ such that:

(i) $\phi_{d,e}^{A_s}(m_{2d+1,s}) \downarrow = 0$, and

(ii) $m_{2d+1,s}$ is B-nonpositive

If there is no such d, set $m_{i,s+1} = m_{i,s}$ for all $i \leqslant s$, and go to the next stage. Otherwise, pick the least such d, and give $m_{2d+1,s}$ the sign $+_B$. In addition, give the sign $-_A$ to each A-nonpositive number used in the computation of $\phi_{d,e}^{A_s}(m_{2d+1,s})$. Finally, reassociate markers as follows. For all $i \leqslant 2d+1$, set $m_{i,s+1} = m_{i,s}$ (i.e. don't move them). For all i such that $d < i \leqslant e$ (if there are any), set $m_{2i,s+1}$ to be the first $e-d$ numbers that are free in A (in increasing order). Don't move any of the odd-indexed markers.

Note: The choice of $d \leqslant i < e$ in the even case and $d < i \leqslant e$ in the odd case is simply to ensure that the index of the markers we move is *above* the index of the chosen marker.

Proof of Success

Let $\mathtt{mov}(d)$ be the maximum number of moves that marker m_d could possibly make throughout the construction. Then:

(a) $\mathtt{mov}(0) = 0$

(b) $\mathtt{mov}(1) = 1$

(c) For $d > 1$, $\mathtt{mov}(d) = \mathtt{mov}(d-1) + \mathtt{mov}(d-2) + 1$

Note that if we pick a least d satisfying the requirements, then we only move markers with indices *strictly above d*. Hence, (a) and (b) are easy to verify. We can verify (c) by induction, but I won't do that here.

Now suppose m_d has permanently stopped moving by stage s. Say WLOG that $d = 2e$, and suppose that $m_{d,s}$ has a $+_A$ at stage s if it ever will. If there is a $+_A$, that means we found that $\phi_{e,s}^{B_s}(m_{d,s}) = 0$. Let $b_1, \ldots, b_n$ be the numbers in $\overline{B_s}$ that were used in the computation of $\phi_{e,s}^{B_s}(m_{d,s})$. So by construction, $b_1, \ldots, b_n$ all have the sign $-_B$.

The worry is that, at some point, one of the b_i's might be given the sign $+_B$ at some later stage t; in which case $\phi_{e,s}^{B}(m_{d,s})$ could disagree with $\phi_{e,s}^{B_s}(m_{d,s})$, since their respective oracles could give different answers to "Do you have b_i?". So we need to make sure that $\phi_{e,s}^{B}(m_{d,s})$ doesn't disagree with $\phi_{e,s}^{B_s}(m_{d,s})$ about whether each b_i is in the oracle (i.e. we need to ensure that their computations are the same on input $m_{d,s}$).

Suppose, for *reductio*, that b_i is given the sign $+_B$ at a later stage t, and let's say $b_i = m_{k,t}$ (where k is odd). If $k < d$, then m_d would be moved at stage t, contrary to hypothesis. So $k > d$. But then at stage s, m_k was placed after all the $-_B$ signs at stage s; so m_k could never have been placed on b_i after s, $\perp$. Hence, if m_d has stopped moving, then no b_i could ever obtain a $+_B$, and so $\phi_{e,s}^{B_s}(m_{d,s}) = \phi_{e,s}^{B}(m_{d,s}) = \phi_{e}^{B}(m_{d,s})$.

Hence, if $m_{d,s}$ does have a $+_A$ sign, that means $\phi_e^{B_s}(m_d) = \phi_e^{B}(m_d) = 0$, satisfying R_d. If $m_{d,s}$ doesn't have a $+_A$ sign, it means $\phi_e^B(m_d) \uparrow$ or $\phi_e^B(m_d) \downarrow \neq 0$, in which case R_d is still satisfied by $m_{d,s}$. Hence, since each marker eventually stops moving, each R_d will be satisfied eventually. Furthermore, the process is clearly r.e.[4] □

[4] It's not recursive, since m_d may not move $\mathtt{mov}(d)$ times, or $\phi_d^S(m_d)$ may be undefined unbenownst to the computer.

3. Let T be a theory of $\langle \mathbb{Z}, < \rangle$.

   (a) Show that T is finitely axiomatizable.

   (b) Show that T has a prime model.

   (c) Show that T has a countable saturated model.

▶ **Answer (a):** Let U be the theory of discrete linear orders without endpoints. U is finitely axiomatizable, and clearly $U \subseteq T$. We'll show that $T \subseteq U$, and hence $T = U$. We'll do this by showing that U is complete (see Marker [4, p. 56–57]).

Claim (1): The models of U are isomorphic to models of the form $\langle L \times \mathbb{Z}, < \rangle$, where $L \neq \varnothing$ is a linear order,[a] and where $<$ is interpreted lexicographically.

[a] The linear order L doesn't have to be a discrete linear order itself. Any ol' linear order will do.

▶ **Proof (1):** Clearly, $\mathcal{A} \cong \langle L \times \mathbb{Z}, < \rangle \Rightarrow \mathcal{A} \models U$. To show the converse, let $\mathcal{A} \models U$, and let $\sim$ be an equivalence relation over $\mathcal{A}$ such that $a \sim b$ iff a is only a finite distance away from b. Each $\sim$-equivalence class $[\![a]\!]$ is isomorphic to $\langle \mathbb{Z}, < \rangle$. So it suffices to show that the $\sim$-equivalence classes are linearly ordered. Define the relation $\ll$ between $\sim$-equivalence classes such that $d \ll e$ iff for all $a \in d$ and $b \in e$, $a < b$.

Reflexivity: Clearly $[\![a]\!] \not\ll [\![a]\!]$. ✓

Asymmetry: Suppose $[\![a]\!] \ll [\![b]\!]$. Then for every $a' \in [\![a]\!]$ and every $b' \in [\![b]\!]$, $a' < b'$. But since $<$ is a linear order, $b' \not< a'$, and hence $[\![b]\!] \not\ll [\![a]\!]$. ✓

Transitivity: Suppose $[\![a]\!] \ll [\![b]\!]$ and $[\![b]\!] \ll [\![c]\!]$. Let $a' \in [\![a]\!]$ and $c' \in [\![c]\!]$. Then $a' < b'$ for $b' \in [\![b]\!]$. Since $b' < c'$, $a' < c'$. Since a' and c' were arbitrary, $[\![a]\!] \ll [\![c]\!]$. ✓

Hence, $\ll$ is a linear order, and so $\mathcal{A} \cong \langle L \times \mathbb{Z}, < \rangle$. □

Claim (2): For every $\mathcal{A} \models U$, $\mathcal{A} \equiv \langle \mathbb{Z}, < \rangle$.

► **Proof (2):** Let $\mathcal{A} := \langle L \times \mathbb{Z}, < \rangle$, and let $\mathcal{M} := \langle \mathbb{Z}, < \rangle$. Let $|a - b|$ denote the distance between a and b (where $|a - b| = \infty$ if a and b are on different $\mathbb{Z}$-chains).

We'll show that $\exists$ (player II) has a winning strategy for the back and forth game $G_n(\mathcal{A}, \mathcal{M})$ for each $n \in \omega$. The worry in any given n-game with $\forall$ (player I) is that if $\forall$ chooses two elements from A in different $\mathbb{Z}$-chains, then he has infinitely many elements between them to choose from, whereas regardless of which elements from $\mathbb{Z}$ that $\exists$ replies with, she'll only have finitely many to choose from. In such a scenario, $\forall$ could drive $\exists$ into a corner until she runs out of options to choose from.

To ensure she avoids "being cornered", $\exists$ will have to make sure that her choice of elements are spread out enough so that if $\forall$ does decide to try and corner her, $\exists$ will always have a way to respond, at least until the game ends.

This can be achieved if, for all $k \leqslant n$, the following requirement is satisfied after round k:

R_k: If $a_1 < \cdots < a_k$ are the elements from A that have been played, and $m_1 < \cdots < m_k$ are the elements that have been played from M ($= \mathbb{Z}$), then for all $i \leqslant k$, either

(i) both $|a_{i+1} - a_i| > 3^{n-k}$ and $|m_{i+1} - m_i| > 3^{n-k}$, or

(ii) $|a_{i+1} - a_i| = |m_{i+1} - m_i| \leqslant 3^{n-k}$.

<u>**Note**</u>: I don't think it will work if the base is 2, but it should work for base 3 or higher.

Suppose R_k is satisfied after round k, and $k < n$. Then if two elements from A, say a_i and a_{i+1} are on different $\mathbb{Z}$-chains, then there's still $3^{n-k} \geqslant 3$ many elements between m_i and m_{i+1}, so $\forall$ can pick any of the elements between a_i and a_{i+1} and $\exists$ will at least be able to respond. If $k = n$, then we'll have a sequence $a_1 < \cdots < a_n$ mapped to $m_1 < \cdots < m_n$, so our map will be a partial embedding. So it suffices to show how these conditions can be satisfied.

We proceed by induction. On round 1, it doesn't matter which elements the players choose. On round 2, if $\forall$ chooses an element whose distance from the element picked in round 1 is no more than 3^{n-2} elements away, then $\exists$ can respond by picking an element the same distance away (in the same direction). Otherwise, $\exists$ can respond by picking an element exactly 3^{n-2} away (in the same direction).

Now, suppose we've ensured each R_k is satisfied for some $k < n$. Let $a_1 < \cdots < a_k$ be the elements from A that have been played, and $m_1 < \cdots < m_k$ be the elements from $\mathbb{Z}$ that have been played. So by the inductive hypothesis, for $1 \leqslant i \leqslant k$, either $|a_{i+1} - a_i| > 3^{n-k} < |m_{i+1} - m_i|$ or $|a_{i+1} - a_i| = |m_{i+1} - m_i| \leqslant 3^{n-k}$.

First, let's suppose $\forall$ chooses a new element $a \in A$ to play. If $a < a_1$, and if $|a_1 - a|$ is finite, $\exists$ can play $m = m_1 - |a - a_1|$ to satsify R_{k+1}. If instead $|a_1 - a|$ is infinite, $\exists$ can play $m = m_1 - 3^n$ to satisfy R_{k+1}. Similarly if $a > a_k$.

If instead, for some $i < k$, $a_i < a < a_{i+1}$, then there are two cases to consider:

Case 1: $|a_{i+1} - a_i| \leqslant 3^{n-k}$. By inductive hypothesis, $|m_{i+1} - m_i| = |a_{i+1} - a_i|$. So then $\exists$ can play $m = m_i + |a - a_i|$, which satisfies R_{k+1}. ✓

Case 2: $|a_{i+1} - a_i| > 3^{n-k}$. By inductive hypothesis, $|m_{i+1} - m_i| > 3^{n-k}$ as well. There are three subcases to consider, depending on how far away a is placed with respect to a_i and a_{i+1}.

Subcase i: $|a - a_i| \leqslant 3^{n-(k+1)}$. Then $|a_{i+1} - a| > 3^{n-(k+1)}$. But since $|m_{i+1} - m_i| > 3^{n-k}$, $\exists$ can play $m = m_i + |a - a_i|$. This will put $|m - m_i| = |a - a_i| \leqslant 3^{n-(k+1)}$ and $|m_{i+1} - m| > 3^{n-(k+1)}$, so R_{k+1} is satisfied. ✓

Subcase ii: $|a_{i+1} - a| \leqslant 3^{n-(k+1)}$. The strategy is similar to the above, except now $m = m_{i+1} - |a_{i+1} - a|$. ✓

Subcase iii: $|a - a_i| < 3^{n-(k+1)}$ and $|a_{i+1} - a| > 3^{n-(k+1)}$. Since $|m_{i+1} - m_i| > 3^{n-k}$, split the regions into thirds, each of size at least $3^{n-(k+1)} = 3^{n-k-1}$. Then if $\exists$ places m somewhere in the middle region, $|m - m_i| > 3^{n-(k+1)}$ and $|m_{i+1} - m| > 3^{n-(k+1)}$. So again, R_{k+1} is satisfied. ✓

If $\forall$ instead chose an element $m \in \mathbb{Z}$ to play, then it's only easier for $\exists$ to choose elements, since she may have infinitely options instead (but she can follow the same basic strategy). Thus, we've shown how $\exists$ can win $G_n(\mathcal{A}, \mathcal{M})$, building a partial embedding with $a_i \mapsto m_i$. Hence, $\mathcal{A} \equiv \mathcal{M}$. □

Hence, U is complete, so $U = T$. □

► **Answer (b):** By Claim (2) above, $\langle \mathbb{Z}, < \rangle$ can be elementarily embedded into any model of T via this elementary chain method. □

▶ **Answer (c):** Let $L_n(x, y)$ be the formula which says, "There are at least n-things between x and y (with $x < y$)," and let $M_n(x, y)$ say, "There are at most n-things between x and y (with $x < y$)." Define:

$$\begin{aligned}\Sigma_+ &:= \{\varphi(\overline{x}) \mid \varphi \text{ is atomic}\} \cup \{L_n(x, y), M_n(x, y) \mid n \in \omega\} \\ \Sigma_- &:= \{\neg\varphi(\overline{x}) \mid \varphi \text{ is atomic}\} \cup \{\neg L_n(x, y), \neg M_n(x, y) \mid n \in \omega\} \\ \Sigma &:= \Sigma_+ \cup \Sigma_-\end{aligned}$$

Claim: Σ is an elimination set for T.

It will suffice to check that $\exists y \bigwedge_{i=1}^{n} \psi_i(\overline{x}, y)$, where $\psi_i \in \Sigma$, is equivalent modulo T to a boolean combination of formulae from Σ. But note that, WLOG, we may assume that ψ_i doesn't have the form:

- "$y = x_k$": otherwise, just replace every instance of y with x_k and remove the existential quantifier. Then we're done.
- "$x_k < x_j$" or "$\neg(x_k < x_j)$": otherwise, we may pull this formula outside the scope of the existential quantifier, and deal with the reduced existential instead.
- "$x_k \neq y$": otherwise, we may replace this formula with "$((x_k < y) \vee (y < x_k))$," and then use distributivity to pull the disjunction outside the scope of the existential quantifier. Then we can deal with the reduced existential instead.
- "$\neg L_n(x_k, y)$" or "$\neg L_n(y, x_k)$": in the first case, we may replace this formula with "$((y = x_k) \vee (y < x_k) \vee M_{n-1}(x_k, y))$," and then use distributivity to pull the disjunction outside the scope of the existential quantifier. Similarly for the second case.
- "$\neg M_n(x_k, y)$" or "$\neg M_n(y, x_k)$": in the first case, we may replace this formula with "$((y = x_k) \vee (y < x_k) \vee L_{n+1}(x_k, y))$," and then use distributivity to pull the disjunction outside the scope of the existential quantifier. Similarly for the second case.

▶ **Proof:** Given the above, it suffices to check formulae of this form (I assume that the indicies are appropriately bounded below, and that the $k_c, l_d, \ldots$ are also appropriately indexed; it would be messy to write out in full detail):

$$\exists y \left(\bigwedge_a (x_a < y) \wedge \bigwedge_b (y < x_b) \wedge \bigwedge_c L_{k_c}(x_c, y) \wedge \right.$$
$$\left. \bigwedge_d L_{l_d}(y, x_d) \wedge \bigwedge_e M_{n_e}(x_e, y) \wedge \bigwedge_f M_{m_f}(y, x_f) \right)$$

Assuming that none of these big conjuncts inside the existential quantifier is empty, we may replace the above formula with the following formula:

$$\bigwedge_{a,b} (x_a < x_b) \wedge \bigwedge_{a,d} L_{l_d+1}(x_a, x_d) \wedge \bigwedge_{a,e} (x_e < x_a \to M_{n_e-1}(x_e, x_a)) \wedge$$
$$\bigwedge_{a,f} L_1(x_a, x_f) \wedge \bigwedge_{b,c} L_{k_c+1}(x_c, x_b) \wedge \bigwedge_{b,e} L_1(x_e, x_b) \wedge$$
$$\bigwedge_{b,f} (x_b < x_f \to M_{m_f-1}(x_b, x_f)) \wedge \bigwedge_{c,d} L_{k_c+l_d+1}(x_c, x_d) \wedge$$
$$\bigwedge_{\substack{c,e \\ k_c < n_e}} (x_e < x_c \to M_{k_c-n_e-1}(x_e, x_c)) \wedge \bigwedge_{\substack{c,e \\ n_e < k_c}} L_{k_c-n_e-1}(x_c, x_e) \wedge$$
$$\bigwedge_{c,f} L_{k_c+1}(x_c, x_f) \wedge \bigwedge_{d,e} L_{l_d+1}(x_e, x_d) \wedge$$
$$\bigwedge_{\substack{d,f \\ l_d < m_f}} (x_d < x_f \to M_{m_f-l_d-1}(x_d, x_f)) \wedge \bigwedge_{\substack{d,f \\ m_f < l_d}} L_{l_d-m_f-1}(x_f, x_d) \wedge$$
$$\bigwedge_{e,f} [L_1(x_e, x_f) \wedge M_{n_e+m_f+1}(x_e, x_f)]$$

If any of these are empty, you'll need to remove the appropriate conjuncts. □

Consider, now, the structure $\mathcal{A} := \langle \mathbb{Q} \times \mathbb{Z}, < \rangle$. We claim that $\mathcal{A}$ is a countably saturated model of T. To show this, it suffices to show that $\mathcal{A}$ realizes every 1-type over finite parameters. Let $X \subseteq A$ be finite, and WLOG assume $b_1 < \cdots < b_n$ completely lists X.

Since Σ is an elimination set, any formula of the form $\varphi(x, b_i)$ is equivalent to some boolean combination of:

- $x < b_i$
- $x > b_i$
- $x = b_i$
- $L_n(x, b_i)$ for $n \in \omega$
- $L_n(b_i, x)$ for $n \in \omega$
- $M_n(x, b_i)$ for $n \in \omega$
- $M_n(b_i, x)$ for $n \in \omega$

Hence, the only complete 1-types over X are those which are (consistent with T and the fact that $b_1 < \cdots < b_n$ and) completions of:

Case i: $\{x = b_i\}$ **for some** $1 \leqslant i \leqslant n$**.** $\mathcal{A}$ realizes this trivially. ✓

Case ii: $\{L_n(x, b_i), M_n(x, b_i)\}$ **for some** $n \in \omega$ **and some** $1 \leqslant i \leqslant n$**.** Since b_i is on a copy of $\mathbb{Z}$, $\mathcal{A}$ realizes this by the unique $(n+1)^{\text{th}}$ elemment before b_i. ✓

Case iii: $\{L_n(b_i, x), M_n(b_i, x)\}$ **for some** $n \in \omega$ **and some** $1 \leqslant i \leqslant n$**.** Since b_i is on a copy of $\mathbb{Z}$, $\mathcal{A}$ realizes this by the unique $(n+1)^{\text{th}}$ elemment after b_i. ✓

Case iv: $\{L_n(x, b_1) \mid n \in \omega\}$**.** $\mathcal{A}$ realizes this with any element of in a $\mathbb{Z}$-chain strictly before b_1's $\mathbb{Z}$-chain. ✓

Case v: $\{L_n(b_n, x) \mid n \in \omega\}$**.** $\mathcal{A}$ realizes this with any element of in a $\mathbb{Z}$-chain strictly after b_n's $\mathbb{Z}$-chain. ✓

Case vi: $\{L_n(b_i, x), L_n(x, b_{i+1}) \mid n \in \omega\}$ **for some** $1 \leqslant i < n$**.** Since $\mathbb{Q}$ is dense, between any two $\mathbb{Z}$-chains in $\mathcal{A}$, there will be another $\mathbb{Z}$-chain, so $\mathcal{A}$ realizes this with any element in a $\mathbb{Z}$-chain between b_i's and b_{i+1}'s $\mathbb{Z}$-chain. ✓

Hence, $\mathcal{A}$ realizes every 1-type over X, so $\mathcal{A}$ is countably saturated (since of course, $\mathbb{Q} \times \mathbb{Z}$ is countable). □

4. Let $f : \mathbb{N} \to \mathbb{N}$ be a total recursive function. Show that there is a formula $\varphi(x)$ in the language of PA such that:

   (i) $\mathsf{PA} \vdash \varphi(\underline{n})$ for each $n \in \mathbb{N}$, and

   (ii) for all n, $f(n) < g(n)$, where $g(n)$ is the least gödel number of a proof of $\varphi(\underline{n})$ in PA.

► **ANSWER:** We start with a proof for a modified Fixed Point Lemma for formulae:

CLAIM (1): For any formula $\psi(x, y)$ in $\mathcal{L}_{\mathsf{PA}}$, there is an formula $\varphi(x)$ in $\mathcal{L}_{\mathsf{PA}}$ such that for each $n \in \mathbb{N}$, $\mathsf{PA} \vdash \varphi(\underline{n}) \leftrightarrow \psi(\underline{n}, \ulcorner\varphi(\underline{n})\urcorner)$.

► **PROOF (1):** Define the (primitive) recursive functions:

$$\mathsf{sub}_2(v, n) = \begin{cases} \ulcorner\theta(\underline{n}, y)\urcorner & \text{if } v = \ulcorner\theta(x, y)\urcorner \\ 0 & \text{otherwise} \end{cases}$$

$$\mathsf{sub}_1(v, n) = \begin{cases} \ulcorner\theta(\underline{n})\urcorner & \text{if } v = \ulcorner\theta(x)\urcorner \\ 0 & \text{otherwise} \end{cases}$$

Since these functions are recursive, PA strongly represents these functions, with say the Δ_1^0-formulae $\mathsf{sub}_2^*(v, n, u)$ and $\mathsf{sub}_1^*(v, n, u)$ respectively. Now, define:

$$\alpha(x, v) := \exists u\, \exists w\ \ (\mathsf{sub}_2^*(v, x, u) \wedge \mathsf{sub}_1^*(u, v, w) \wedge \psi(x, w))$$

Finally, define $\varphi(x) := \alpha(x, \ulcorner\alpha(v_1, v_2)\urcorner)$. Then:

$$\begin{aligned} \mathsf{PA} \vdash \varphi(\underline{n}) &\leftrightarrow \alpha(\underline{n}, \ulcorner\alpha\urcorner) \\ &\leftrightarrow \exists u\, \exists w\ \ [\mathsf{sub}_2^*(\ulcorner\alpha\urcorner, \underline{n}, u) \wedge \mathsf{sub}_1^*(u, \ulcorner\alpha\urcorner, w) \wedge \psi(\underline{n}, w)] \\ &\leftrightarrow \exists w\ \ [\mathsf{sub}_1^*(\ulcorner\alpha(\underline{n}, v_2)\urcorner, \ulcorner\alpha\urcorner, w) \wedge \psi(\underline{n}, w)] \\ &\leftrightarrow \psi(\underline{n}, \ulcorner\alpha(\underline{n}, \ulcorner\alpha\urcorner)\urcorner) \\ &\leftrightarrow \psi(\underline{n}, \ulcorner\varphi(\underline{n})\urcorner) \end{aligned}$$

where $\ulcorner\alpha\urcorner = \ulcorner\alpha(v_1, v_2)\urcorner$. □

Now, since f is recursive, PA can weakly represent the graph of f with a Σ^0_1-formula, say $G_f(x,y)$. The goal, then, is to devise a formula $\psi(x,v)$ which says "The sentence coded by v is not provable in a proof whose gödel number is less than or equal to $f(x)$". Then, by the modified Fixed Point Lemma above, we can construct a sentence which says "I am not provable in a proof of size less than or equal to $f(n)$".

We let this formula be:

$$\psi(x,v) := \exists y\ \exists s\ [G_f(x,y) \wedge s > y \wedge \forall t\ < s \neg \mathsf{Prf}_{\mathsf{PA}}(t,v)]$$

By Claim (1), there is a formula $\varphi(x)$ where $\mathsf{PA} \vdash \varphi(\underline{n}) \leftrightarrow \psi(\underline{n}, \ulcorner\varphi(\underline{n})\urcorner)$. Furthermore, notice that $\psi(x,v)$ is Σ^0_1, and so $\varphi(x)$ is as well.

Claim (2): For each $n \in \mathbb{N}$, $\mathsf{PA} \vdash \varphi(\underline{n})$.

▶ **Proof (2):** Suppose $\mathsf{PA} \nvdash \varphi(\underline{n})$ for some n. Then certainly, $\varphi(\underline{n})$ is not provable in a proof of size less than or equal to $f(n)$. Hence, $\varphi(\underline{n})$ is true. But since $\varphi(\underline{n})$ is Σ^0_1, and since PA proves all true Σ^0_1, $\mathsf{PA} \vdash \varphi(\underline{n})$, $\bot$. □

Since PA is Σ^0_1-sound, and since $\mathsf{PA} \vdash \varphi(\underline{n})$ for each $n \in \mathbb{N}$, it follows that each $\varphi(\underline{n})$ must be true, i.e. they're provable but not in a proof of size less than or equal to $f(n)$. Hence, $f(n) < g(n)$ for all n. □

5. (a) Show that $a \in M$ is definable iff for every elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$, and every automorphism $\sigma : N \to N$, σ fixes a, i.e. $\sigma(a) = a$.

   (b) Show by example (with proof) that it may happen that a is not definable but every automorphism $\sigma : M \to M$ fixes a.

▶ **Answer (a):**

($\Rightarrow$) Suppose $a \in M$ is definable by $\varphi(x)$. Let $\mathcal{N} \succcurlyeq \mathcal{M}$. Thus, $\mathcal{N} \models \forall x\ (\varphi(x) \leftrightarrow x = a)$. Then if $\sigma : N \to N$ is an automorphism, we have that $\mathcal{N} \models \forall x\ (\varphi(x) \leftrightarrow x = \sigma(a))$, in which case $\mathcal{N} \models \forall x\ (x = a \leftrightarrow x = \sigma(a))$, i.e. $\sigma(a) = a$. □

($\Leftarrow$) Suppose $a \in M$ is not definable. Let:

$$T := \mathsf{ElDiag}(\mathcal{M}) \cup \{a \neq c\} \cup \{\varphi(c) \mid \mathcal{M} \models \varphi(a)\}$$

where a is a constant from $\mathsf{ElDiag}(\mathcal{M})$, and c is a new constant.

Claim: T is finitely satisfiable.

▶ **Proof:** Suppose that some finite subset $T_0 \subseteq T$ is not satisfiable. Then for some $\varphi_1(c), \ldots, \varphi_n(c) \in T_0$, we have that $\mathsf{ElDiag}(\mathcal{M}) \vdash \bigwedge_{i=1}^{n} \varphi_i(c) \rightarrow a = c$. Since c doesn't occur in $\mathsf{ElDiag}(\mathcal{M})$, $\mathsf{ElDiag}(\mathcal{M}) \vdash \forall x\ (\bigwedge_{i=1}^{n} \varphi_i(x) \rightarrow x = a)$. But then by construction, $\bigwedge_{i=1}^{n} \varphi_i(x)$ would define a, since we have $\mathcal{M} \models \forall x\ (x = a \rightarrow \bigwedge_{i=1}^{n} \varphi_i(x))$, $\perp$. □

Thus, by Compactness, T is satisfiable. Let $\mathcal{N} \models T$. Then by construction, $\mathcal{M} \preccurlyeq \mathcal{N}$. Furthermore, there is a $c^{\mathcal{N}} = b \in N$ where $b \neq a$ satisfies the same type as a. Hence (by Marker [4, Lemma 4.1.5, p. 117]), there is an elementary extension of $\mathcal{N}$ (and hence of $\mathcal{M}$) such that there is an automorphism of this extension σ satisfying $\sigma(a) = b$. □

▶ **Answer (b):** <u>Note</u>: There are many examples of these kinds of structures. I have provided several examples for illustration.

- Let $\mathcal{L} = \{c_i \mid i \in \omega\}$. Consider a model $\mathcal{M}$ in which every object is named by a distinct constant except for one lonely element, a. Then the only things you can say in this model are what things are or are not named by this or that constant. And since formulae are finite, and since there are infinitely many constants, a won't be definable. But automorphisms must preserve the assignment of constants, and hence $\mathcal{M}$ only has the trivial automorphism. □
- Let $\mathcal{L} = \{<\}$. Consider $\langle \omega_1, < \rangle$. Recall there are no non-trivial automorphisms of any ordinal (otherwise, well-ordering would guarantee that for any non-trivial automorphism σ, there's a least α for which $\sigma(\alpha) \neq \alpha$. $\sigma(\alpha)$ can't be below α, since for $\beta < \alpha$, $\sigma(\beta) = \beta$.

So $\sigma(\alpha) > \alpha$. But σ must preserve the order, so σ can't send anything to β for $\alpha < \beta < \sigma(\alpha)$, which means σ isn't surjective, $\perp$). Furthermore, since there are only countably-many formulae and uncountably-many elements in ω_1, it follows that there must be undefinable elements. $\square$

- Let $\mathcal{L} = \{0, 1, +, \cdot\}$. Consider $\mathbb{R}$ as our model. We can define:
  - Subtraction: $x - y :=\ \iota z(z + y = x)$.
  - Negative: $-x :=\ \iota z(x + z = 0)$.
  - Order: $x < y := \exists u\, \exists v\ (u \neq 0 \wedge u = v \cdot v \wedge (y - x) = u)$
  - Rationals: if $q = m/n \in \mathbb{Q}$, where $m, n \in \mathbb{Z}$, then $q :=\ \iota z(\underline{n} \cdot z = \underline{m})$ (where $\underline{\pm k} := \pm(1 + \cdots + 1)$, k-times).

  By part (a) and the above, every automorphism of $\mathbb{R}$ must fix $\mathbb{Q}$. Automorphisms must also preserve the ordering, since they preserve all formulae. Hence, if $r \in \mathbb{R}$, and $\sigma(r) = r + \epsilon$ (for instance), then the rationals between r and $r + \epsilon$ have to be moved above r in order to preserve the order; but σ must also fix them, $\perp$. Hence, $\mathbb{R}$ has no non-trivial automorphism.

  However, by part (a), if we can find an elementary extension of $\mathbb{R}$ with non-trivial automorphisms, then $\mathbb{R}$ will have some undefinable elements. In fact, $\mathbb{C}$ will do, since we can have automorphisms which don't fix transcendentals. $\square$

6. Let $E := \{e \in \omega \mid W_e = \{x \in \omega \mid \exists y\ \in \omega\, (y + y = x)\}\}$. Compute the position of E in the arithmetic hierarchy.

▶ **Answer:** E is the index set for the even numbers. Hence:

$$\begin{aligned} e \in E &\Leftrightarrow W_e = \{n \in \omega \mid n \text{ is even}\} \\ &\Leftrightarrow \forall n\ \ (n \in W_e \leftrightarrow n \text{ is even}) \\ &\Leftrightarrow \forall n\ \ (n \in W_e \leftrightarrow \exists m\ < n\, (m + m = n)) \\ &= \forall n\ \ (\Sigma^0_1 \leftrightarrow \Delta^0_1) \\ &= \Pi^0_2 \end{aligned}$$

To show that E is Π_2^0-hard, we'll show that $\mathsf{Tot} \leqslant_m E$. Let:

$$f(e,x) = \begin{cases} 1 & \text{if } \phi_e(x/2)\downarrow \text{ where } x \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

This function is recursive, so by *s-m-n*, there is a total recursive $s(x)$ such that $f(e,x) = \phi_{s(e)}(x)$. So then:

$$\begin{aligned} e \in \mathsf{Tot} &\Rightarrow \forall x\ (\phi_e(x)\downarrow) \Rightarrow \forall x\ (f(e,2x)\downarrow) \Rightarrow \forall x\ (\phi_{s(e)}(2x)\downarrow) \Rightarrow s(e) \in E \\ e \notin \mathsf{Tot} &\Rightarrow \exists x\ (\phi_e(x)\uparrow) \Rightarrow \exists x\ (f(e,2x)\uparrow) \Rightarrow \exists x\ (\phi_e(2x)\uparrow) \quad \Rightarrow s(e) \notin E \end{aligned}$$

Hence $e \in \mathsf{Tot}$ iff $s(e) \in E$, which completes the reduction. □

7. Show that for any complete theory T having infinite models, there is a model $\mathcal{M} \vDash T$ and a descending chain of elementary submodels $\mathcal{M}_\alpha \vDash T$ (where $\alpha \in \omega+1$) so that $\mathcal{M} = \mathcal{M}_0 \succ \mathcal{M}_1 \succ \mathcal{M}_2 \succ \cdots \succ \mathcal{M}_\omega = \bigcap_{i=0}^{\infty} \mathcal{M}_i$ and $\mathcal{M} \cong \mathcal{M}_\alpha$ for all $\alpha \in \omega + 1$.

▶ **Answer:** Let T^* be the Skolemization of T, and take $\mathcal{M}$ to be the Ehrenfeucht-Mostowski model whose spine is $\mathbb{Q} \cup \{+\infty, -\infty\}$. Define $\mathcal{A}'_q$ to be $\operatorname{Hull}(\{c_r \mid |r| \leqslant |q|\})$. Consider the chain of models where $\mathcal{M}_0 := \mathcal{M}$, $\mathcal{M}_i := \mathcal{A}'_{1+1/i}$ for $0 < i < \omega$, and $\mathcal{M}_\omega := \bigcap_i \mathcal{M}_i$. We need to check the following:

- **For each $\alpha \in \omega + 1$, $\mathcal{M}_\alpha \vDash T$.** Each $\mathcal{M}_\alpha$ is the Skolem hull of some subset of M_0. Since T^* has Skolem functions, it has a universal axiomatization, and universally axiomatized theories are preserved from superstructures to substructures. Hence, $\mathcal{M}_\alpha \vDash T$. ✓
- **For each $i, j \in \omega + 1$, where $i < j$, $\mathcal{M}_i \succ \mathcal{M}_j$.** Suppose $i \neq 0$. Then create an elementary embedding $\mathfrak{f}_{j,i} : \mathcal{M}_j \to \mathcal{M}_i$ as follows. First, have $\mathfrak{f}_{j,i}(c_0) = c_0$. Next, have $\mathfrak{f}_{j,i}(c_{\pm(1+1/j)}) = c_{\pm(1+1/i)}$. Finally, if $n/(1+1/j) = m$, then have $\mathfrak{f}_{j,i}(c_n) = c_{m\cdot(1+1/i)}$. The density $[-(1+1/i), (1+1/i)]$ ensures we will be able to find a match for any c_n with $n \in [-(1+1/j), (1+1/j)]$.

  For $i = 0, j > 1$, we can compose maps, so we just need to consider $i = 0, j = 1$. As before, have $\mathfrak{f}_{1,0}(c_{\pm 2}) = c_{\pm\infty}$, and have

$\mathfrak{f}_{1,0}(c_0) = c_0$. For everything else in between, simply scale n appropriately. The density of $[-\infty, +\infty]$ ensures we'll always be able to find a match for c_n with $n \in [-2, 2]$. ✓

- **For each $\alpha \in \omega + 1$, $\mathcal{M} \cong \mathcal{M}_\alpha$.** An isomorphism on linear orders induces an isomorphism on the structures with those linear orders as spines. Since any closed interval can be put into isomorphsim with $\mathbb{Q} \cup \{\pm\infty\}$, we get that $\mathcal{M} \cong \mathcal{M}_\alpha$ for $\alpha \in \omega + 1$. ✓ □

8. Let $\mathcal{L}$ be a language having at least one constant symbol. Let T be a consistent $\mathcal{L}$-theory which is universally axiomatized. Prove that the following two conditions are equivalent:

AP: T has the amalgamation property: For any three models $\mathcal{M}, \mathcal{A}, \mathcal{B} \models T$, with $\mathcal{M} \subseteq \mathcal{A}, \mathcal{B}$, there is a model $\mathcal{D} \models T$ and embeddings $f : A \to D$, $g : B \to D$, such that $f \upharpoonright M = g \upharpoonright M$.

QFI: T satisfies quantifier-free interpolation, i.e. for any (disjoint) $\overline{x}, \overline{y}, \overline{z}$, and quantifier-free formulae $\varphi(\overline{x}, \overline{y})$ and $\psi(\overline{y}, \overline{z})$, if $T \vdash \varphi(\overline{x}, \overline{y}) \to \psi(\overline{y}, \overline{z})$, then there is a quantifier-free formula $\theta(\overline{y})$ so that $T \vdash \varphi(\overline{x}, \overline{y}) \to \theta(\overline{y})$ and $T \vdash \theta(\overline{y}) \to \psi(\overline{y}, \overline{z})$.[5]

▶ **Answer (AP ⇒ QFI):** Suppose $T \vdash \varphi(\overline{x}, \overline{y}) \to \psi(\overline{y}, \overline{z})$, where $\varphi(\overline{x}, \overline{y})$ and $\psi(\overline{x}, \overline{y})$ are quantifier-free. If $T \vdash \neg\varphi(\overline{x}, \overline{y})$, then $\theta = \bot$ will suffice. Similarly, if $T \vdash \psi(\overline{y}, \overline{z})$, then $\theta = \top$ will suffice. So assume $T \nvdash \neg\varphi(\overline{x}, \overline{y})$ and $T \nvdash \psi(\overline{y}, \overline{z})$. Let $\overline{a}, \overline{b}, \overline{c}$ all be new constants added to the language, and consider the theory:

$$\Gamma := T \cup \{\varphi(\overline{a}, \overline{b})\} \cup \left\{ \neg\theta(\overline{b}) \;\middle|\; \begin{array}{l} \theta \text{ is quantifier-free and} \\ T \vdash \theta(\overline{b}) \to \psi(\overline{b}, \overline{c}) \end{array} \right\}$$

Since T is universally axiomatized, T will prove that a negated instantiation of one of these axioms implies anything, and this negated instantiation will be quantifier-free, so the rightmost set is nonempty.

Suppose Γ is not satisfiable. That means for some $\theta_1(\overline{y}), \ldots, \theta_n(\overline{y})$ (all quantifier-free), we have $T \vdash \bigwedge_{i=1}^{n} \neg\theta_i(\overline{b}) \to \neg\varphi(\overline{a}, \overline{b})$, i.e. we have

[5] "$T \vdash \sigma(\overline{v})$" is short for "$T \vdash \forall\overline{v}\, \sigma(\overline{v})$", where $\sigma(\overline{v})$ is a formula

$T \vdash \varphi(\bar{a}, \bar{b}) \to \bigvee_{i=1}^{n} \theta_i(\bar{b})$. But we already have $T \vdash \theta_i(\bar{b}) \to \psi(\bar{b}, \bar{c})$ for $1 \leqslant i \leqslant n$, and thus $T \vdash \bigvee_{i=1}^{n} \theta_i(\bar{b}) \to \psi(\bar{b}, \bar{c})$. Hence, $\bigvee_{i=1}^{n} \theta_i(\bar{y})$ can be our quantifer-free interpolant. So it suffices to show that:

CLAIM: Γ is unsatisfiable.

▶ **PROOF:** Suppose for *reductio* that Γ is satisfiable. Let $\mathcal{N} \models \Gamma$. Consider $\mathcal{M} := \langle \bar{b} \rangle_{\mathcal{N}}$ and $\mathcal{A} := \langle \bar{a}, \bar{b} \rangle_{\mathcal{N}}$. First, note $\mathcal{M} \subseteq \mathcal{A} \subseteq \mathcal{N}$. Next, since T is universally axiomatized, T is preserved going into substructures, so $\mathcal{M}, \mathcal{A} \models T$. Furthermore, $\mathcal{A} \models \varphi(\bar{a}, \bar{b})$, since we had $\mathcal{N} \models \varphi(\bar{a}, \bar{b})$ and since $\varphi(\bar{x}, \bar{y})$ is quantifier-free. Finally, we have that $\mathcal{M} \models \neg\theta(\bar{b})$ for every quantifier-free $\theta(\bar{y})$ such that $T \vdash \theta(\bar{y}) \to \psi(\bar{y}, \bar{z})$.

Now, let:

$$\Sigma := T \cup \mathsf{Diag}(\mathcal{M}) \cup \{\neg\psi(\bar{b}, \bar{c})\}$$

Given this last statement, we claim:

CLAIM: Σ is satisfiable.

▶ **SUBPROOF:** Suppose not. Then by Compactness, for some (literal) sentences $\sigma_1(\bar{b}), \ldots, \sigma_n(\bar{b}) \in \mathsf{Diag}(\mathcal{M})$, we have that $T \vdash \bigwedge_{i=1}^{n} \sigma_n(\bar{b}) \to \psi(\bar{b}, \bar{c})$. Since for each i, $\sigma_i(\bar{b}) \in \mathsf{Diag}(\bar{b})$, $\bigwedge_{i=1}^{n} \sigma_i(\bar{b}) \in \mathsf{Diag}(\mathcal{M})$. So by construction, $\neg \bigwedge_{i=1}^{n} \sigma_i(\bar{b}) \in \Gamma$, and hence $\mathcal{N} \models \neg \bigwedge_{i=1}^{n} \sigma_i(\bar{b})$. But then $\mathcal{M} \models \neg \bigwedge_{i=1}^{n} \sigma_i(\bar{b})$, since $\mathcal{M} \subseteq \mathcal{N}$, and so $\neg \bigwedge_{i=1}^{n} \sigma_i(\bar{b}) \in \mathsf{Diag}(\mathcal{M})$, $\perp$. □

Take a model $\mathcal{B} \models \Sigma$. Then $\mathcal{M} \subseteq \mathcal{B}$, so by the fact that T has **AP**, there is a $\mathcal{D} \models T$ such that we have $\mathcal{M} \subseteq \mathcal{A} \subseteq \mathcal{D}$ and $\mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{D}$. But since $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are quantifier-free, it follows that $\mathcal{D} \models \varphi(\bar{a}, \bar{b}) \wedge \neg\psi(\bar{b}, \bar{c})$, contrary to the supposition that $T \vdash \varphi(\bar{x}, \bar{y}) \to \psi(\bar{y}, \bar{z})$, $\perp$. □

This completes the proof. □

▶ **ANSWER (QFI ⇒ AP):** The proof is a modification of the proof of the Elementary Amalgamation Theorem (see Hodges [2, p. 135]). We'll see that **QFI** will be exactly the property we need to make this modified proof go through.

Let $\mathcal{M}, \mathcal{A}, \mathcal{B} \models T$, with $\mathcal{M} \subseteq \mathcal{A}, \mathcal{B}$. WLOG, we can arrange things so that $A \cap B = M$. Consider the theory:

$$\Gamma := T \cup \mathsf{Diag}(\mathcal{A}) \cup \mathsf{Diag}(\mathcal{B})$$

Suppose Γ has a model, say $\mathcal{D}^+ \models \Gamma$, with $\mathcal{D} := \mathcal{D}^+ \upharpoonright \mathcal{L}$. Then by construction, $\mathcal{M} \subseteq \mathcal{A}, \mathcal{B} \subseteq \mathcal{D}$. Defining $f(a) = a^{\mathcal{D}}$ and $g(b) = b^{\mathcal{D}}$ for $a \in A$, $b \in B$, we'll have $f \upharpoonright M = g \upharpoonright M$. So it suffices to show that:

CLAIM: Γ is finitely satisfiable.

▶ **PROOF:** Let $\Gamma_0 \subseteq \Gamma$ be finite. Let

$$\varphi(\overline{m}, \overline{a}) := \bigwedge \{\theta(\overline{m}, \overline{a}) \mid \theta(\overline{m}, \overline{a}) \in \Gamma_0 \cap \mathsf{Diag}(\mathcal{A})\}$$

$$\psi(\overline{m}, \overline{b}) := \bigwedge \{\theta(\overline{m}, \overline{b}) \mid \theta(\overline{m}, \overline{b}) \in \Gamma_0 \cap \mathsf{Diag}(\mathcal{B})\}$$

where $\overline{m} \in M$, $\overline{a} \in A - M$, and $\overline{b} \in B - M$. Suppose Γ_0 has no model. Then $T \vdash \varphi(\overline{m}, \overline{a}) \to \neg\psi(\overline{m}, \overline{b})$. Since T doesn't ever mention $\overline{a}, \overline{b}, \overline{m}$, $T \vdash \forall \overline{x}, \overline{y}, \overline{z}\ (\varphi(\overline{x}, \overline{y}) \to \neg\psi(\overline{y}, \overline{z}))$. So by **QFI**, there is some $\theta(\overline{y})$ such that $T \vdash \forall \overline{x}, \overline{y}\ (\varphi(\overline{x}, \overline{y}) \to \theta(\overline{y}))$ and $T \vdash \forall \overline{y}, \overline{z}\ (\theta(\overline{y}) \to \neg\psi(\overline{y}, \overline{z}))$.

Now since $\mathsf{Diag}(\mathcal{A}) \vdash \varphi(\overline{m}, \overline{a})$, and $\mathcal{A} \models T$, we have $\mathcal{A} \models \theta(\overline{m})$. But since θ is quantifier-free, it's preserved going into substructures, so $\mathcal{M} \models \theta(\overline{m})$. Again, since θ is quantifier-free, we have $\mathcal{B} \models \theta(\overline{m})$. But then $\mathcal{B} \models \neg\psi(\overline{m}, \overline{b})$, and since ψ is quantifier-free, we have $\mathsf{Diag}(\mathcal{B}) \vdash \neg\psi(\overline{m}, \overline{b})$, $\perp$. □

So by Compactness, Γ is satisfiable. □

March 2013

1. Let R_e be the e^{th} recursively enumerable subset of $\omega \times \omega$ in some standard enumeration. Let $E := \{e \mid R_e$ is an equivalence relation on $\omega\}$. Give the exact location of E in the standard arithmetic hierarchy, and justify your answer.

▶ **Answer:** The following shows that E is Π^0_2:

$$\begin{aligned} E(n) \Leftrightarrow\ & \forall x\ \neg R_n(x,x) \wedge \forall x\ \forall y\ \ (R_n(x,y) \to R_n(y,x)) \wedge \\ & \forall x\ \forall y\ \forall z\ \ (R_n(x,y) \wedge R_n(y,z) \to R_n(x,z)) \\ =\ & \Pi^0_1 \wedge \Pi^0_2 \wedge \Pi^0_2 \\ =\ & \Pi^0_2 \end{aligned}$$

In order to show that this is optimal, it suffices to show that $\mathsf{Tot} \leqslant_m E$. Define the function:

$$h(e,x,y) = \begin{cases} \langle x,y \rangle & \text{if } x = y \text{ and } \phi_e(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

This function is recursive, so by *s-m-n*, there's a total recursive $s(x)$ such that $h(e,x,y) = \phi_{s(e)}(x,y)$. Then:

$$\begin{aligned} e \in \mathsf{Tot} &\Rightarrow \forall x\ (\phi_e(x)\downarrow) \Rightarrow \forall x\ \ (\langle x,x\rangle \in \mathsf{dom}\,(\phi_{s(e)})) \Rightarrow s(e) \in E \\ e \notin \mathsf{Tot} &\Rightarrow \exists x\ (\phi_e(x)\uparrow) \Rightarrow \exists x\ \ (\langle x,x\rangle \notin \mathsf{dom}\,(\phi_{s(e)})) \Rightarrow s(e) \notin E \end{aligned}$$

Hence $e \in \mathsf{Tot}$ iff $s(e) \in E$, which completes the reduction. □

2. Let T_0 and T_1 be axiomatizable extensions of PA. Suppose that $T_0 \vdash \mathsf{Con}\,(T_1)$ and $T_1 \vdash \mathsf{Con}\,(T_0)$. Show that T_0 and T_1 are inconsistent.

▶ **Answer:** Since both T_0 and T_1 are axiomatizable, and since we have $T_0 \vdash \mathsf{Con}\,(T_1)$, $T_1 \vdash \mathsf{Prv}_{T_0}\,(\ulcorner \mathsf{Con}\,(T_1) \urcorner)$. Now, since $T_1 \vdash \mathsf{Con}\,(T_0)$, $T_1 \vdash \neg \mathsf{Prv}_{T_0}\,(\ulcorner \neg \mathsf{Con}\,(T_1) \urcorner)$. But $T_1 \vdash \neg \mathsf{Con}\,(T_1) \to \mathsf{Prv}_{T_0}\,(\ulcorner \neg \mathsf{Con}\,(T_1) \urcorner)$ (since PA could prove $\neg \mathsf{Con}\,(T_1)$ if it were true), so $T_1 \vdash \mathsf{Con}\,(T_1)$ by *modus tollens*. Hence, by Gödel's second incompleteness theorem, T_1 is inconsistent. A similar argument shows T_0 is inconsistent. □

3. Let T be a univerally axiomatized $\mathcal{L}$-theory. Suppose $T \vdash \forall x\, \exists y\, \theta(x, y)$ where θ is quantifier-free. Show that there is a finite list $t_1, \ldots, t_n$ of $\mathcal{L}$-terms such that $T \vdash \forall x\, \bigvee_{j=1}^{n} \theta(x, t_j(x))$.

▶ **Answer:** Suppose there is no such finite list. Then for any finite list ℓ of (indices of) terms with one free variable, we'll have a model $\mathcal{A}_\ell \models T$ such that for some $a_\ell \in A_\ell$, $\mathcal{A}_\ell \models \bigwedge_{i \in \ell} \neg\theta(a_\ell, t_i(a_\ell))$.

Add a constant c to $\mathcal{L}$, and define the theory:

$$\Gamma := T \cup \{\neg\theta(c, t(c)) \mid t \text{ is an } \mathcal{L} \cup \{c\} \text{-term}\}$$

Let $\Gamma_0 \subseteq \Gamma$ be a finite subset, say with $\neg\theta(c, t_i) \in U$ for $i \in s$ (where s is finite). Then we can take the model $\mathcal{A}_s$ and interpret $c^{\mathcal{A}_s} = a_s$. Hence, Γ_0 is satisfiable. So by Compactness, Γ has a model.

Let $\mathcal{B} \models \Gamma$, and consider $\mathcal{M} = \langle c^{\mathcal{B}} \rangle_{\mathcal{B}}$. Then every element of $\mathcal{M}$ is denoted by some closed term $t(c)$. Since $\neg\theta(x, y)$ is quantifier-free, it's preserved in $\mathcal{M}$, so $\mathcal{M} \models \neg\theta(c, t(c))$ for every term $t(x)$; and since every element is denoted by a closed term $t(c)$, we have that $\mathcal{M} \models \forall y\, \neg\theta(c, y)$. But since T has a universal axiomatization, it's preserved under substructures, so since $\mathcal{B} \models T$, $\mathcal{M} \models T$. And since $T \vdash \forall x\, \exists y\, \theta(x, y)$, $\mathcal{M} \models \forall x\, \exists y\, \theta(x, y)$, which can't be since $\mathcal{M} \models \neg\exists y\, \theta(c, y)$. $\perp$ $\square$

4. For $X \subseteq \mathbb{N}$, we say that $X \in \mathsf{St}(\mathcal{M})$ iff for some formula $\varphi(y, \overline{x})$ and some parameters $\overline{a} \in M$: $k \in X \Leftrightarrow \mathcal{M} \models \varphi[\underline{k}^{\mathcal{M}}, \overline{a}]$.

   (a) Show that for $\mathcal{M} \models \mathsf{PA}$, there is a nonrecursive $A \in \mathsf{St}(\mathcal{M})$.

   (b) Show that for all nonrecursive $A \subseteq \mathbb{N}$, there is a $\mathcal{M} \models \mathsf{PA}$ such that $A \notin \mathsf{St}(\mathcal{M})$.

▶ **Answer (a):** Let A be the set of Π^0_1-formulae $\theta(x)$ such that, for some fixed k, $\mathcal{M} \models \theta(\underline{k})$. A is nonrecursive by Gödel's incompleteness theorems, since $\mathsf{Th}(\mathcal{M})$ is complete. Hence, it suffices to show that $A \in \mathsf{St}(\mathcal{M})$.

The idea will be to show that $\mathcal{M}$ has a nonstandard element a such that a codes the set A. If so, then it follows that $n \in A$ iff $\mathcal{M} \models (a)_n \neq \underline{0}$, and hence $A \in \mathsf{St}(\mathcal{M})$.

With this in mind, define:

$$\varphi(x) := \exists s\ \forall i < x\ \left((s)_i \neq \underline{0} \leftrightarrow \mathsf{Sat}_{\Pi_1^0}(i, \underline{k})\right)$$

That is, $\varphi(x)$ says "There is a sequence of numbers which picks out exactly the Π_1^0-formulae with gödel numbers less than x satisfied by k." Trivially, $\mathsf{PA} \vdash \varphi(\underline{0})$. In addition, $\mathsf{PA} \vdash \forall x\ (\varphi(x) \rightarrow \varphi(x+1))$: if m is a sequence which codes all of the Π_1^0-formulae satisfied by k with gödel numbers less than n, then we can simply extend this sequence either by 0 (in the case where $n+1$ doesn't code a Π_1^0-formula satisfied by k) or by 1 (in the case where it does)—and PA is smart enough to know this. Hence, by the induction scheme, $\mathsf{PA} \vdash \forall x\, \varphi(x)$. But then $\mathcal{M} \models \forall x\, \varphi(x)$, and hence every nonstandard element satisfies φ. Thus, in $\mathcal{M}$, there is an element $a \in \mathcal{M}$ which codes exactly the Π_1^0-formulae that hold of k in $\mathcal{M}$. □

▶ **Answer (b):** Suppose there was a formula $\varphi(x, \overline{b})$ such that $a \in A$ iff $\mathcal{M} \models \varphi(a, \overline{b})$. Using the same strategy as above with induction schema, it follows that $\mathcal{M} \models \forall x\, \exists s\, \forall i < x\, \left((s)_i \neq \underline{0} \leftrightarrow \varphi(i, \overline{b})\right)$. But then there would be a nonstandard $a \in \mathcal{M}$ that coded the set A in $\mathcal{M}$. Hence, it suffices to show that some model lacks a code for A, i.e. the type

$$p(x) := \{(x)_i \neq \underline{0} \mid i \in A\} \cup \{(x)_i = \underline{0} \mid i \notin A\}$$

is omitted in some model. We do this by showing that $p(x)$ is not a principal type.

Suppose $p(x)$ is principal in every model of PA. Say it's supported by $\theta(x, \overline{b})$. Then both $\mathsf{PA} \vdash \forall \overline{x}\ (\theta(x) \rightarrow (x)_i \neq \underline{0})$ for all $i \in A$, and $\mathsf{PA} \vdash \forall x\ (\theta(x) \rightarrow (x)_i = \underline{0})$ for all $i \notin A$. But in that case, we would have a recursive procedure for determining whether or not $a \in A$: just start searching through the proofs of PA until you either find a proof of $\forall x\ (\theta(x) \rightarrow (x)_a \neq \underline{0})$ or of $\forall x\ (\theta(x) \rightarrow (x)_a = \underline{0})$. Hence, if A is nonrecursive, $p(x)$ cannot be principal. □

5. (a) Let $R \subseteq \mathbb{N}^2$ be r.e, and assume that for all n, m, $R(n,m) \Rightarrow W_n \neq W_m$. Show there is an n such that for all m, $\neg R(n,m)$.

(b) Let $S \subseteq \mathbb{N}^4$ be r.e, and assume that for all n, m, p, q, we have that $S(n, m, p, q) \Rightarrow (W_n \neq W_p \vee W_m \neq W_q)$. Show that there are n, m such that for all p, q, $\neg S(n, m, p, q)$.

▶ **Answer (a):** Suppose not, i.e. suppose $\forall n\, \exists m\, R(n, m)$. Define:

$$s(n) := \mu m\, [R(n, m)]$$

This is recursive since R is r.e., and furthermore it's total by our hypothesis. That is, $s(x)$ is a total recursive function such that for all e, $W_e \neq W_{s(e)}$. But this can't be, since by the Recursion Theorem, there must be some e such that $W_e = W_{s(e)}$, $\perp$. □

▶ **Answer (b):** Suppose not. We can redefine R from part (a) as $R(n, m) := S(n, n, m, m)$. So let $s(x)$ be as in part (a). Define two new functions:

$$f(n, m) := s(n)$$
$$g(n, m) := s(m)$$

These are again both total and recursive. And again, for all e, d, we have $W_{f(e,d)} = W_{s(e)} \neq W_e$ and $W_{g(e,d)} = W_{s(d)} \neq W_d$. But by the Double Recursion Theorem, there are some e, d such that $W_{f(e,d)} = W_e$ and $W_{g(e,d)} = W_d$, $\perp$. □

6. Let T be a consistent, decidable theory in the language with one binary relation symbol R, and suppose that all models of T are infinite. Show that T has a model $\mathcal{A} = \langle \omega, R \rangle$ such that the full elementary diagram of $\mathcal{A}$ is recursive.

▶ **Answer:** This problem is just like this problem, page 46, except here you don't need to worry about omitting any types. □

7. Let T be a theory having infinite models. Show that there is a model $\mathcal{A} \models T$ and a collection of (proper) elementary submodels $\mathcal{A}_q \prec \mathcal{A}$ indexed by $q \in \mathbb{Q}$ so that for $q < r$ we have $\mathcal{A}_q \prec \mathcal{A}_r$ and $\mathcal{A}_q \cong \mathcal{A}$.

▶ **Answer:** Take a skolemization of T, T^*, and then consider an Ehrenfeucht-Mostowski model $\mathcal{A} \vDash T^*$ whose spine is $\mathbb{Q}$. Define $\mathcal{A}_q := \operatorname{Hull}(\{c_r \mid r < q\})$. We need to check:

- **For each** $q \in \mathbb{Q}$**,** $\mathcal{A}_q \prec \mathcal{A}$**.** This is immediate, since $\mathcal{A}_q$ is a Skolem hull. ✓
- **For each** $q, r \in \mathbb{Q}$**, whenever** $q < r$**,** $\mathcal{A}_q \prec \mathcal{A}_r$**.** This is also immediate, since the identity map is an elementary embedding. ✓
- **For each** $q \in \mathbb{Q}$**,** $\mathcal{A}_q \cong \mathcal{A}$**.** Since any isomorphism between linear orders induces an isomorphism on structures with those orders as spines, and since there's an isomorphism between $\mathbb{Q}$ and any initial segment of $\mathbb{Q}$, it follows that $\mathcal{A}_q \cong \mathcal{A}$. ✓ □

8. (a) Let U be a nonprincipal ultrafilter on ω. Show that $\{e \mid W_e \in U\}$ is not Δ^0_2.

   (b) Show that there is a nonprincipal ultrafilter U on ω such that $\{e \mid W_e \in U\}$ is Δ^0_3.

▶ **Answer (a):** Suppose U is nonprincipal. Then U cannot contain any finite sets, and must contain every cofinite set. We will show that $A := \{e \mid W_e \in U\}$ is Π^0_2-hard. Let B be a Π^0_2 set, i.e. for some recursive relation $R(x, y, z)$, $B(n) \Leftrightarrow \forall x\, \exists y\, R(x, y, n)$. Define:

$$g(n, s) = \begin{cases} 1 & \text{if } \forall x < s\ \exists y\, R(x, y, n) \\ \uparrow & \text{otherwise} \end{cases}$$

This is recursive, so $g(n, s) = \phi_{t(n)}(s)$ for some total recursive t. Notice that if $g(n, k)\uparrow$, then for all $k' \geqslant k$, $g(n, k')\uparrow$, i.e. g is either total or finite. Now:

$$\begin{aligned} n \in B &\Rightarrow \forall x\, \exists y\, R(x,y,n) \\ &\Rightarrow \forall s\, (g(n,s) = 1) \\ &\Rightarrow \forall s\, (\phi_{t(n)}(s)\downarrow) \\ &\Rightarrow W_{t(n)} = \mathbb{N} \text{ and } \mathbb{N} \in U, \text{ so} \\ &\Rightarrow t(n) \in A \\ n \notin B &\Rightarrow \exists x\, \forall y\, \neg R(x,y,n) \\ &\Rightarrow \exists s\, \forall m \geqslant s\, (g(n,m)\uparrow) \\ &\Rightarrow \exists s\, \forall m \geqslant s\, (\phi_{t(n)}(m)\uparrow) \\ &\Rightarrow W_{t(n)} \notin U, \text{ since } W_{t(n)} \text{ is finite} \\ &\Rightarrow t(n) \notin A \end{aligned}$$

Hence, $n \in B \Leftrightarrow t(n) \in A$, which completes the reduction.[6] □

► **Answer (b):** We construct a descending chain $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ as follows. First, set $A_0 := \mathbb{N}$. Next, given A_n, check whether $A_n \cap W_n$ is infinite. If it is, set $A_{n+1} := A_n \cap W_n$. Otherwise, set $A_{n+1} := A_n$.

Let U_n be the set of sets that are approximately A_n (i.e. they're identical modulo finitely many things). Define $U := \bigcup_{i=0}^{\omega} U_i$. We need to check that:

- U **is in fact an ultrafilter.** We accounted for all Boolean combinations of sets in U since we enumerated all of the r.e sets. ✓
- U **is nonprincipal.** We included all cofinite sets in U_0. ✓
- U **is** Δ^0_3**.** By checking the complexity of our construction, we notice that at each stage, we only need to know whether $A_n \cap W_n$ is finite. This can be checked by Inf, and hence the whole process is recursive in $0''$, i.e. the whole process is Δ^0_3. ✓ □

[6] A similar reduction can be used to show that A is also Σ^0_2-hard, in which case A is Δ^0_3-hard. I haven't worked through all of the details, but I don't think there's any major roadblocks.

9. Let $\langle A, < \rangle$ be a countable dense linear order without endpoints. Let $\mathsf{St}_1(\mathcal{A})$ be the set of 1-types of $\langle A, <, a \rangle_{a \in A}$. For $p, q \in \mathsf{St}_1(\mathcal{A})$, we say $p \sim q$ iff there is an automorphism $\alpha : A \to A$ for which:

$$p = \alpha_*(q) = \{\varphi(x, \alpha(b_1), \ldots, \alpha(b_n)) \mid \varphi(x, b_1, \ldots, b_n) \in q\}$$

Show that there are 6 equivalence classes.

August 2012

1. Let $E \subseteq \mathbb{N}^2$ be an r.e equivalence relation.

   (a) Show that if E has finitely many equivalence classes, then E is recursive.

   (b) Prove or refute: if each of E's equivalence classes is finite, then E is recursive.

► **Answer (a):** Suppose E has k-many equivalence classes with $e_1, \ldots, e_k$ as representatives of each equivalence class. To determine if a tuple $\langle a, b\rangle \in E$, start looking through the enumeration of E for $\langle a, b\rangle$, and simultaneously start looking for the pair of tuples $\langle a, e_i\rangle$ and $\langle e_j, b\rangle$ for some $1 \leqslant i, j \leqslant k$ (you're guaranteed to find one eventually). If you find $\langle a, b\rangle$, or if you find this pair of tuples so that $i = j$, then $\langle a, b\rangle \in E$. Otherwise, you'll find this pair of tuples so that $i \neq j$, so $\langle a, b\rangle \notin E$. This procedure is algorithmic, so E is recursive. □

► **Answer (b):** The claim is false. Let A be a Σ^0_1 set, and start enumerating the elements of A as $a_1, a_2, a_3, \ldots$ (with no repeats). We will build an E which is r.e and whose equivalence classes are all finite, but that's not recursive.

Stage 0**:** Set $E_0 = \{\langle 0, 0\rangle\}$.

Stage 1**:** Set $E_1 = E_0 \cup \{\langle a_1, a_1\rangle\}$.

Stage $2s$**:** Set $E_{2s} = E_{2s-1} \cup \{\langle s, s\rangle\}$. This ensures reflexivity is met.

Stage $2s+1$**:** Suppose E_{2s} has already been constructed. By design, the first $m := \frac{s(s+1)}{2}$ elements of A have been put in equivalence classes of incremental size, with the largest class being of size s. So take the next $s+1$-many elements $a_{m+1}, \ldots, a_{m+s+1}$, and put them together in an equivalence class. Enumerate all of the (finitely many) ordered-pair combinations of this equivalence class, and add them to E_{2s} to make E_{2s+1}.

This process is clearly r.e, and furthermore each equivalence class only has finitely many elements (either just one or s-many elements). However, it's not recursive. Suppose neither $n, m \in A$. Then clearly the tuple $\langle n, m\rangle \notin E$. But at no stage of our construction can this be concluded, since we can't be certain that further in the construction we'll find either n or m in the next $\frac{s(s+1)}{2}$ elements of A. Hence, we don't have an effective procedure for determining whether $\langle n, m\rangle \in E$.[a] $\square$

[a] However, in the case where one of $n, m \in A$, we can tell by just seeing if the other appears in the equivalence class whenever the first shows up in the enumeration.

2. (a) Let T be a theory in a countable language having infinitely many 1-types. Show that T has, up to isomorphism, more than one countable model.

   (b) Give an example of a theory T with just one 1-type, but infinitely many countable models. Prove that your example works.

▶ **Answer (a):** This is essentially one part of the Engeler-Ryll-Nardzewski-Svenonius Theorem [see 2, Theorem 6.3.1]. We'll prove the contrapositive, i.e. we'll show that if T is $\aleph_0$-categorical, then T has only finitely many 1-types. This will be done in two steps.

Claim (1): If T is $\aleph_0$-categorical, then every 1-type over T is principal.

▶ **Proof (1):** Suppose for *reductio* that T is $\aleph_0$-categorical, but the 1-type $p(x)$ over T is not principal. Then by the Omitting Types Theorem, there's a model $\mathcal{A}' \models T$ that omits p. By Downward Löwenheim-Skolem, we can find a countable $\mathcal{A} \preccurlyeq \mathcal{A}'$ that also omits p. But since p is a type, there is a model $\mathcal{B}' \models T$ that realizes it, say with $b \in B'$. Again by Downward Löwenheim-Skolem, there is a countable $\mathcal{B} \preccurlyeq \mathcal{B}'$ containing b, and hence realize p. But then $\mathcal{A}$ and $\mathcal{B}$ are two countable nonisomorphic models of T, $\perp$. $\square$

Claim (2): If every 1-type over T is principal, then there are only finitely many 1-types over T.

▶ **Proof (2):** Suppose for *reductio* that every 1-type over T is principal, but there are infinitely many 1-types over T. Let $\theta_1(x), \theta_2(x), \theta_3(x), \ldots$ be the generators of these 1-types. Add a new constant c to the language and define:

$$\Gamma := T \cup \{\neg\theta_i(c) \mid i \in \omega\}$$

Take a finite $\Gamma_0 \subseteq \Gamma$. Then Γ_0 only mentions finitely many generators $\theta_1(x), \ldots, \theta_n(x)$. Hence, we can let c denote an object satisfying $\theta_{n+1}(x)$ for instance, and so Γ_0 will be satisfiable. By Compactness, Γ is satisfiable, which means there's another 1-type not generated by any of the θ_i's, $\perp$. □

Hence, if T is $\aleph_0$-categorical, then there are at most finitely many 1-types over T. □

▶ **Answer (b):** Let $\mathcal{L} = \langle s \rangle$, where s is a unary function symbol, whose intended interpretation is the successor function. We'll use "$s^n(x)$" as an abbreviation for s applied n-times to x (with "$s^0(x)$" being just x). Let T be the theory containing the following axioms:

- $\forall x\, \exists! y\, (s^n(y) = x)$ for all $n \in \omega$
- $\forall x, y\, (s^n(x) = s^n(y) \leftrightarrow s^m(x) = s^m(y))$ for all $m, n \in \omega$
- $\forall x\, (s^n(x) \neq x)$ for all $n \in \omega - \{0\}$

T has as a model $\langle \mathbb{Z}, s \rangle$, so T is consistent. Furthermore, any model with countably many isolated $\mathbb{Z}$-chains will also be a countable model of T. Hence, T has infinitely many countable models.

Notice that for any $a \in M$ where $\mathcal{M} \models T$, then the substructure containing a and all its successors and predecessors will generate a $\mathbb{Z}$-chain. We'll say that a $c \in M$ is in a's $\mathbb{Z}$-chain if for some k, either $\mathcal{M} \models s^k(c) = a$ or $\mathcal{M} \models s^k(a) = c$.

To show that T only has one 1-type, it suffices to show that for any two elements $a, b \in M$ where $\mathcal{M} \models T$, there is an automorphism σ on $\mathcal{M}$ sending $a \mapsto b$. In that case, $\mathcal{M} \models \varphi(a)$ iff $\mathcal{M} \models \varphi(\sigma(a))$ iff $\mathcal{M} \models \varphi(b)$, which means a and b satisfy the same 1-type.

To prove this, I'll need to show the following claim:

CLAIM: T has quantifier-elimination.

▶ **PROOF (CLAIM):** We need to reduce the following formula (where the indices are appropriately bound):

$$\exists x \left[\bigwedge_a s^{n_a}(x) = y_a \wedge \bigwedge_b s^{m_b}(z_b) = x \wedge \right.$$
$$\left. \bigwedge_c s^{k_c}(x) \neq u_c \wedge \bigwedge_d s^{l_d}(v_d) \neq x \right]$$

There are three cases to consider.

Case i: The big b-conjunct is non-empty. Then pick a z_b and replace every instance of x with $s^{m_b}(z_b)$. ✓

Case ii: The big b-conjunct is empty, but the big a-conjunct is non-empty. Then pick the y_a with the least n_a. Let e be the index for that y_a. Rewrite the formula as follows:

$$\bigwedge_a s^{n_a - n_e}(y_e) = y_a \wedge \bigwedge_{\substack{c \\ k_c < n_e}} y_e \neq s^{n_e - k_c}(u_c) \wedge$$
$$\bigwedge_{\substack{c \\ k_c \geqslant n_e}} s^{k_c - n_e}(y_e) \neq u_c \wedge \bigwedge_d s^{l_d + n_e}(v_d) \neq y_e$$

This will suffice. ✓

Case iii: Both the big a-conjunct and big b-conjunct are empty. Then rewrite the formula as $\bigwedge_{c,d} s^{k_c + l_d}(v_d) \neq u_c$. ✓

Either way, we've elminated the quantifiers. □

Let $a, b \in M$ for some $\mathcal{M} \models T$. There are three cases to consider.

Case 1: $\mathcal{M} \models s^n(a) = b$ for some $n \in \omega$. Then define σ as follows. If $c \in M$ is such that either $\mathcal{M} \models s^k(c) = a$ or $\mathcal{M} \models s^k(a) = c$ for some

k, then send $c \mapsto s^n(c)$. Otherwise, send $c \mapsto c$. (In other words, σ shifts a's $\mathbb{Z}$-chain, and leaves everything else alone.)

Now, take $c, d \in M$ and $i, j \in \omega$. We want to show that $\mathcal{M} \models s^i(c) = s^j(d)$ iff $\mathcal{M} \models s^i(\sigma(c)) = s^j(\sigma(d))$. There are four subcases to consider:

Subcase i: Neither c nor d are in a's $\mathbb{Z}$-chain. Then $\mathcal{M} \models s^i(c) = s^j(d)$ iff $\mathcal{M} \models s^i(\sigma(c)) = s^j(\sigma(d))$. ✓

Subcase ii: c is in a's $\mathbb{Z}$-chain, but d isn't. Then $\mathcal{M} \models s^i(c) \neq s^j(d)$. Hence, $\mathcal{M} \models s^{n+i}(c) \neq s^j(d)$, so $\mathcal{M} \models s^i(\sigma(c)) \neq s^j(\sigma(d))$. ✓

Subcase iii: d is but c isn't. Similar to Subcase iii. ✓

Subcase iv: Both c and d are in a's $\mathbb{Z}$-chain. Then $\mathcal{M} \models s^i(c) = s^j(d)$ iff $\mathcal{M} \models s^{i+n}(c) = s^{j+n}(d)$ iff $\mathcal{M} \models s^i(\sigma(c)) = s^j(\sigma(d))$.

In either subcase, σ preserves all the quantifier-free formula, so by quantifier-elimination, σ is an automorphism. ✓

Case 2: $\mathcal{M} \models s^n(b) = a$ for some $n \in \omega$. Similar to Case 1, except shift in the other direction. ✓

Case 3: Neither of the above. Then define σ as follows. Let $c \in M$. If $\mathcal{M} \models s^n(b) = c$, then map $c \mapsto s^n(a)$. If $\mathcal{M} \models s^n(c) = b$, then map $c \mapsto \ \imath y\,[s^n(y) = a]$. Similarly in the case where a and b are switched. Otherwise, map $c \mapsto c$. (In other words, σ swaps a's s-chain and b's $\mathbb{Z}$-chain.)

Now, take $c, d \in M$ and $i, j \in \omega$. Again, we want to show that $\mathcal{M} \models s^i(c) = s^j(d)$ iff $\mathcal{M} \models s^i(\sigma(c)) = s^j(\sigma(d))$. There are now five subcases to consider:

Subcase i: c and d are both in a's $\mathbb{Z}$-chain. Then c and d are mapped to elements the same distance apart, so we have that $\mathcal{M} \models s^i(c) = s^j(d)$ iff $\mathcal{M} \models s^i(\sigma(c)) = s^j(\sigma(d))$. ✓

Subcase ii: c is in a's $\mathbb{Z}$-chain, but d is in b's $\mathbb{Z}$-chain. So $\mathcal{M} \models s^i(c) \neq s^j(d)$. Then c gets sent to an element in b's $\mathbb{Z}$-chain, and d gets sent to element in a's $\mathbb{Z}$-chain, so $\mathcal{M} \models s^i(\sigma(c)) \neq s^j(\sigma(d))$. ✓

Subcase iii: d is in a's $\mathbb{Z}$-chain, but c is in b's $\mathbb{Z}$-chain. Similar to Subcase ii. ✓

Subcase iv: c and d are both in b's $\mathbb{Z}$-chain. Similar to Subcase i. ✓

Subcase v: Neither c nor d are in either a's or b's $\mathbb{Z}$-chain. Then $\sigma(c) = c$ and $\sigma(d) = d$, so $\mathcal{M} \models s^i(c) = s^j(d)$ iff $\mathcal{M} \models s^i(\sigma(c)) = s^j(\sigma(d))$. ✓

Again, σ preserves quantifier-free formula, so by quantifier-elimination, σ is an automorphism. ✓

Thus, T only has one 1-type. $\square$

3. Let $A := \{\langle e, n\rangle \mid |W_e| = n\}$.

   (a) Show that A is Δ^0_2.

   (b) Show that A is not Σ^0_1 or Π^0_1.

▶ **Answer (a):** A is Σ^0_2 as shown by the equivalence below.

$$\begin{aligned}\langle e, n\rangle \in A \Leftrightarrow\ & |W_e| = n \\ \Leftrightarrow\ & \exists s\, [\mathsf{Seq}(s) \land \mathsf{lh}(s) = n \ \land \forall i, j < n\ ((s)_i \neq (s)_j) \land \\ & \quad \forall i < n\ ((s)_i \in W_e) \land \forall m\ (\forall i < n\ ((s)_i \neq m) \rightarrow (m \notin W_e))]\end{aligned}$$

To show that $\overline{A}$ is Σ^0_2:

$$\begin{aligned}\langle e, n\rangle \notin A \Leftrightarrow\ & (|W_e| = m \land m \neq n) \text{ or } |W_e| = \aleph_0 \\ \Leftrightarrow\ & \exists s, m\ [\mathsf{Seq}(s) \land \mathsf{lh}(s) = m \land n \neq m \land \forall i, j < n\ ((s)_i \neq (s)_j) \land \\ & \quad \forall i < n\ ((s)_i \in W_e) \land \forall m\ (\forall i < n\ ((s)_i \neq m) \rightarrow (m \notin W_e))] \\ \lor\ & \exists s, m\ [\mathsf{Seq}(s) \land \mathsf{lh}(s) = m \land n < m \land \forall i, j < m\ ((s)_i \neq (s)_j) \land \\ & \quad \forall i < m\ ((s)_i \in W_e)]\end{aligned}$$

Hence, A is Δ^0_2. $\square$

▶ **Answer (b):** If A were Σ^0_1, then the following function would be recursive:

$$f(e,x) = \begin{cases} n & \text{if } \langle e,n\rangle \in A \\ \uparrow & \text{otherwise} \end{cases}$$

That is, if W_e is finite, $f(e,x)\downarrow$ and $f(e,x) = |W_e|$ (x here is a dummy variable). By s-m-n, there's a total recursive $s(x)$ such that $f(e,x) = \phi_{s(e)}(x)$. By the Recursion Theorem, there is some fixed point d such that $\phi_{s(d)} = \phi_d$.

Consider W_d. If W_d is finite, then $f(d,x) = |W_d|$ for all x, in which case the domain of $\phi_{s(d)}$ is total (and so infinite). But since $\phi_{s(d)} = \phi_d$, that means W_d is infinite, $\perp$. Hence, W_d must be infinite. So $f(d,x)\uparrow$ for all x. But then $\phi_{s(d)}\downarrow$ is undefined for all x as well. And since $\phi_{s(d)} = \phi_d$, $W_d = \varnothing$, $\perp$. Hence, f cannot be recursive, and so A cannot be Σ^0_1.[7]

If A were Π^0_1, then let $A(e,n)$ iff $\forall z\ R(e,n,z)$, for some recursive $R(x,y,z)$. The following function would then be recursive:

$$g(e,n,x) = \begin{cases} 1 & \text{if } \forall z \leqslant x\ R(e,n,z) \\ \uparrow & \text{otherwise} \end{cases}$$

By s-m-n, there's a total recursive s such that $g(e,n,x) = \phi_{s(e,n)}(x)$. By the Double Recursion Theorem, there exists a,b such that $W_{s(a,b)} = W_a$ and $W_{s(a,b)} = W_b$ (using s for both functions; though we really only need the first fixed point).

Consider W_a. If W_a is finite, then $g(a,b,x)\downarrow$ for all x, and hence $W_{s(a,b)} = W_a = \mathbb{N}$, $\perp$. Hence, W_a is infinite. But then, for some x, $\neg R(a,b,x)$, and hence for all $z \geqslant x$, $\neg R(a,b,z)$. Thus, $g(a,b,x)\downarrow$ for only finitely many x if any. But then $W_{s(a,b)} = W_a$ is finite, $\perp$. Hence, g must not be recursive, and so A is not Π^0_1. $\square$

4. Let T be a theory with infinite models. Show that there is a chain of models $(\mathcal{A}_i \mid i < \omega)$ of T such that for each $i \in \omega$, $\mathcal{A}_{i+1} \prec \mathcal{A}_i$.

[7] This problem, page 50, is a similar problem, though the proof is roughly the same.

> ▶ **ANSWER:** Take the skolemization of a completion of T. Build an Ehrenfeucht-Mostowski model $\mathcal{A}$ from the order-indiscernibles $\{a_q \mid q \in \mathbb{Q}\}$. Let $\mathcal{A}_q$ be the Skolem hull of $\{a_r \mid r < q\}$. Then the chain of models $\mathcal{A}_1 \succ \mathcal{A}_{1/2} \succ \mathcal{A}_{1/3} \succ \cdots$ works. See this problem, page 31, for more details. □

5. Show that there is a total function $f : \mathbb{N} \to \mathbb{N}$ such that:

- f is strictly increasing
- the range of f is r.e, and
- whenever $g : \mathbb{N} \to \mathbb{N}$ is total and recursive, then $\exists n\ (g(n) < f(n))$.

> ▶ **ANSWER:** Impossible: there is no such function f.
>
> > ▶ **PROOF:** Suppose f is as above. Let h be the recursive enumeration of $\operatorname{ran}(f)$ (that lists $\operatorname{ran}(f)$ without repetitions, for simplicity). Define g by induction so that $g(n+1) = 1 + \max(h(0), \ldots, h(n), g(n))$. Clearly, this is total since h is total. But there can't be an n such that $g(n) < f(n)$; since f is strictly increasing, $f(n)$ is the n^{th} smallest member of $\operatorname{ran}(f)$, so $\max(h(0), \ldots, h(n-1)) \geqslant f(n)$, in which case we're guaranteed that $g(n) > f(n)$, $\perp$. □
>
> **Note**: If the range is co-r.e instead, then this is possible, but I have yet to fill in the details. . .

6. Let T be a theory of $\langle \mathbb{Z}, s \rangle$, where $s(n) = n + 1$ for all $n \in \mathbb{Z}$.

   (a) Show that T is recursively axiomatizable.

   (b) Show that T is not finitely axiomatizable.

   (c) Describe the countable models of T. How many are there, up to isomorphism?

   (d) Describe the models of T of size κ, where κ is uncountable. How many are there, up to isomorphism?

▶ **Answer (a):** We'll use "$s^n(x)$" as an abbreviation for s applied n-times to x (with "$s^0(x)$" being just x). Let U be the theory containing the axioms:

- $\forall x\, \exists! y\, (s^n(y) = x)$ for all $n \in \omega$
- $\forall x, y\, \ (s^n(x) = s^n(y) \leftrightarrow s^m(x) = s^m(y))$ for all $m, n \in \omega$
- $\forall x\, (s^n(x) \neq x)$ for all $n \in \omega - \{0\}$

This is the same theory used in this problem, page 36, part (b). Clearly, $\langle \mathbb{Z}, s\rangle \vDash U$, so $U \subseteq T$. To show that $T \subseteq U$, it suffices to show that U is complete.

This is easily done. First, from our proof in part (b), here, page 38, we showed that U has quantifier-elimination. Furthermore, we gave a computable method for determining the quantifier-free equivalent of a given formula. Second, notice that because the only nonlogical symbol is a function symbol, the only quantifier-free sentences are $\top$ and $\bot$. Hence, by quantifier-elimination, we have an effective decision procedure for determining whether a sentence φ is equivalent to $\top$ or $\bot$. But this implies that for any sentence φ, either $U \vdash \varphi$ or $U \vdash \neg\varphi$. □

▶ **Answer (b):** Suppose T was finitely axiomatizable. Let φ be the conjunction of these axioms, and let U be as given above.

Claim: $U \cup \{\neg\varphi\}$ is satisfiable.

▶ **Subproof:** If φ is an axiomatization of T, then φ must imply that there are no s-loops. Hence, $\neg\varphi$ must be consistent with there being s-loops (of any finite size, since the lack of s-loops of any given size is not inconsistent).

So take a finite subset $U' \subseteq U \cup \{\neg\varphi\}$. Then there will be a maximum k for which U' says "There are no s-loops of length k." Since the existence of s-loops are consistent with $\neg\varphi$, taking the model of $k+1$ many things with a s-loop of length $k+1$ will model U'. So by Compactness, $U \cup \{\neg\varphi\}$ is satisfiable. □

Hence, $U \nvdash \varphi$, so $T \nvdash \varphi$, $\bot$. □

▶ **Answer (c):** First:

Claim: Every $\mathcal{A} \models T$ is isomorphic to $\langle \lambda \times \mathbb{Z}, s \rangle$ for some λ.

▶ **Proof:** Let $\mathcal{A} \models T$. Say that $a, b \in A$ are **s-linked** if for some $k \in \omega$, either $\mathcal{A} \models s^k(a) = b$ or $\mathcal{A} \models s^k(b) = a$. Let λ be the length of the longest sequence of elements from A such that none are s-linked to any of the other elements in the sequence. Let $\kappa := |\lambda \times \mathbb{Z}| = \max(\omega, \lambda)$, and let $\mathcal{B} := \langle \lambda \times \mathbb{Z}, s \rangle$. We'll produce a winning strategy for $\exists$ in the game $G_\kappa(\mathcal{A}, \mathcal{B})$.

On round 0, it doesn't matter which elements are picked. Let's suppose now that we've completed round $\gamma < \kappa$, and that $a_0, \ldots, a_\gamma$ are the elements from A that have been chosen, while $b_0, \ldots, b_\gamma$ are the elements of B that have been chosen.

Suppose on round γ^+, $\forall$ plays $a \in A$. There are three cases:

Case 1: $s^k(a_i) = a$ for some $k \in \omega$ and some $i \in \gamma^+$. Then $\exists$ looks for the b_i that was picked on round i, and plays $s^k(b_i)$. This preserves the successor function. ✓

Case 2: $s^k(a) = a_i$ for some $k \in \omega$ and some $i \in \gamma^+$. Similarly, $\exists$ plays the element $c \in B$ such that $s^k(c) = b_i$. ✓

Case 3: Neither of the above. Then $\exists$ searches for an element $c \in \lambda$ which is not the first coordinate of any b_i, and plays $\langle c, 0 \rangle$. By definition of λ, this can only happen λ-many times, so $\exists$ will always be able to find such a c when this happens. And since $\langle c, 0 \rangle$ isn't s-linked to any other b_i, this preserves the successor function. ✓

Next, suppose instead $\forall$ chooses a $b \in B$. Again, $\exists$ runs through the three subcases above. If b is s-linked to any of the b_i, then $\exists$ plays as in Case 1 or 2. Otherwise, $\exists$ searches for an a not s-linked to any of the a_is played so far. Again, this is always possible, since there are exactly λ-many such b_is where this will occur.

Thus, at the end of the game, we'll have constructed a bijection from A to $\alpha \times \mathbb{Z}$ that preserves the successor function. Hence, our bijection will be an isomorphism. □

Note that clearly, if $\mathcal{A} \cong \langle \lambda \times \mathbb{Z}, s \rangle$, then $\mathcal{A} \models T$. Hence, models of the form $\langle \lambda \times \mathbb{Z}, s \rangle$ are exactly the models of T.

By the above claim, if $\mathcal{A} \models T$ is countable, then $\mathcal{A} \cong \langle \lambda \times \mathbb{Z}, s \rangle$ for some $\lambda \leqslant \omega$. Hence, there are at most countably-many countable models of T. In fact, there are exactly $\aleph_0$-many. For suppose $n, m \leqslant \omega$ and $n < m$. Then if one tries to build an isomorphism from $n \times \mathbb{Z}$ to $m \times \mathbb{Z}$, one has to preserve the assignment of the successor function. Hence, each of the n-many $\mathbb{Z}$-chains from $n \times \mathbb{Z}$ must be sent to n-many distinct $\mathbb{Z}$-chains in $m \times \mathbb{Z}$, leaving some $\mathbb{Z}$-chains in $m \times \mathbb{Z}$ left out. Hence, $\langle n \times \mathbb{Z}, s \rangle \ncong \langle m \times \mathbb{Z}, s \rangle$. □

▶ **Answer (d):** If $\kappa > \aleph_0$, then the above argument shows how to construct an isomorphism between any two models of T of size κ. Hence, T is κ-categorical. □

7. Let $\varphi(v)$ be a formulae in the language of PA.

   (a) Suppose φ is Σ^0_1, and $\mathsf{PA} \vdash \exists v\, \varphi(v)$. Show that $\mathsf{PA} \vdash \varphi(\underline{n})$ for some n.

   (b) Give an example of a formula $\varphi(v)$ such that $\mathsf{PA} \vdash \exists v\, \varphi(v)$, but for all $n \in \mathbb{N}$, $\mathsf{PA} \nvdash \varphi(\underline{n})$.

   (c) Suppose φ is Σ^0_1 and that $T \vdash \exists v\, \varphi(v)$, where $T \supseteq \mathsf{PA}$ is consistent. Does it follow that $T \vdash \varphi(\underline{n})$ for some n?

▶ **Answer (a):** Since PA is Σ^0_1-sound, $\exists v\, \varphi(v)$ must be true on $\mathbb{N}$. Hence, there must be some $n \in \mathbb{N}$ such that $\varphi(n)$ is true. Furthermore, since every model of PA is an end extension of $\mathbb{N}$, it follows that every model of PA contains n, and hence satisfies $\varphi(n)$. So $\mathsf{PA} \vdash \varphi(\underline{n})$. □

▶ **Answer (b):** Let $\varphi(v)$ be the following formula:

$$\varphi(v) := (v = 0 \leftrightarrow \mathsf{Con}(\mathsf{PA})) \wedge (\mathsf{Prf}_{\mathsf{PA}}(v, \ulcorner \bot \urcorner) \leftrightarrow \neg \mathsf{Con}(\mathsf{PA}))$$

First, we show that $\mathsf{PA} \vdash \exists v\, \varphi(v)$. Reasoning in PA, suppose $\mathsf{Con}(\mathsf{PA})$.

Then setting $v = 0$ satsifies the formula above, since by definition $\mathsf{Con}(\mathsf{PA}) \to \forall x\, \neg\mathsf{Prf}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$; hence if $\mathsf{Con}(\mathsf{PA})$, then $\exists v\, \varphi(v)$. Suppose instead $\neg\mathsf{Con}(\mathsf{PA})$. Then by shifting quantifiers, we can let v just be the proof of $\bot$. Hence, either way, there exists such a v so that $\varphi(v)$. So now outside of PA again, we conclude $\mathsf{PA} \vdash \exists v\, \varphi(v)$.

Second, we show that for all $n \in \mathbb{N}$, $\mathsf{PA} \nvdash \varphi(\underline{n})$. If there were such a n, then it couldn't be 0, since in that case PA would prove $\mathsf{Con}(\mathsf{PA})$ via $\varphi(\underline{n})$. So it would have to be the case that $n \neq 0$. But then $\mathsf{PA} \vdash \mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner) \leftrightarrow \neg\mathsf{Con}(\mathsf{PA})$. But PA can decide whether the LHS is true, since $\mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner)$ is Δ^0_1. Hence, either $\mathsf{PA} \vdash \mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner)$ or $\mathsf{PA} \vdash \neg\mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner)$. If it proves the latter, it's inconsistent, since $\mathsf{PA} \vdash \mathsf{Con}(\mathsf{PA})$ (via $\varphi(\underline{n})$). If it proves the former, then since $\mathsf{PA} \vdash \varphi(\underline{n})$, $\mathsf{PA} \vdash \neg\mathsf{Con}(\mathsf{PA})$; and since PA only proves true Σ^0_1, PA would be inconsistent. Hence, there cannot be an n such that $\mathsf{PA} \vdash \varphi(\underline{n})$. □

▶ **Answer (c):** It does not follow. Take $T = \mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$. By definition, $T \vdash \exists v\, \mathsf{Prf}_{\mathsf{PA}}(v, \ulcorner\bot\urcorner)$. Now suppose there were an n such that $T \vdash \mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner)$. Since T is consistent, T is Π^0_1-sound. And since $\mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner)$ is Δ^0_1 (and so Π^0_1), it would follows that $\mathsf{Prf}_{\mathsf{PA}}(\underline{n}, \ulcorner\bot\urcorner)$ is true. But then PA is inconsistent, and hence T is inconsistent, contrary to supposition. Hence, taking $\varphi(v) := \mathsf{Prf}_{\mathsf{PA}}(v, \ulcorner\bot\urcorner)$ provides us with a counterexample. □

8. Let $\mathcal{L}$ be the language with one binary relation symbol R, and no other non-logical symbols. Let T be a consistent, decidable $\mathcal{L}$-theory. Let Σ be a non-principal 1-type of T such that Σ is decidable. Show that T has a model $\mathcal{A} = \langle A, R^{\mathcal{A}} \rangle$ that omits Σ and is such that A and $R^{\mathcal{A}}$ are both recursive.

▶ **Answer:** The idea is to run through the construction of the Omitting Types Theorem, using just the 1-type $\Sigma(x)$. Then, we'll check that this procedure is decidable at every step.

First, enumerate the formulae of $\Sigma(x)$ as $\sigma_0(x), \sigma_1(x), \sigma_2(x), \ldots$. Add countably many new Henkin constants $c_0, c_1, c_2, \ldots$ to the language, and enumerate the sentences of this expanded language as $\varphi_0, \varphi_1, \varphi_2, \ldots$. In what follows, let $\Delta_n \coloneqq \Gamma_n - T$ (so that we can talk about the finitely many sentences in $\Gamma_n - T$).

Stage 0**:** Set $\Gamma_0 = T$.

Stage $n.1$**:** Having constructed Γ_n, check to see if $\Gamma_n \cup \{\varphi_n\}$ is consistent. If it is, set $\Gamma'_n = \Gamma_n \cup \{\varphi_n\}$; otherwise, we set $\Gamma'_n = \Gamma_n$. This can be done in a decidable way since T is deciable and Δ_n is finite (i.e. we can search for a proof of $T \vdash \bigwedge \Delta_n \to \neg\varphi_n$).

Stage $n.2$**:** If $\Gamma_n \neq \Gamma'_n$ and if $\varphi_n = \exists x\, \theta$, for some $\theta(x)$, then pick the first unused Henkin constant c_j and let $\Gamma''_n = \Gamma'_n \cup \{\theta(c_j)\}$; otherwise set $\Gamma''_n = \Gamma'_n$. This can be done in a decidable way, since Δ'_n is finite and none of the c's occur in T.

Stage $n.3$**:** Begin the search for a $\sigma_i(x)$ such that $\Gamma''_n \nvdash \sigma_i(c_n)$. This is always possible because Σ is nonprincipal, for that means that there cannot be a formula $\psi(x)$ such that $T \vdash \forall x\ (\psi(x) \to \sigma_i(x))$ for every $i \in \omega$, and in particular, there can't be a formula $\psi(x)$ such that $T \vdash [\bigwedge \Delta''_n \wedge \psi(c_n)] \to \sigma_i(c_n)$ for all $i \in \omega$. Hence, for some $i \in \omega$, we must have $\Gamma''_n \nvdash \sigma_i(c_n)$. Furthermore, since T and $\Sigma(x)$ are decidable, we have a decidable procedure that will guarantee we find one such σ_i eventually. Pick the first such σ_i and set $\Gamma_{n+1} = \Gamma''_n \cup \{\neg\sigma_i(c_n)\}$.

Finally, set $\Gamma = \bigcup_{i\in\omega} \Gamma_i$. By construction, Γ is both consistent and complete. Thus, we can use Downward Löwenheim-Skolem to obtain a countable model $\mathcal{A} \models \Gamma$ generated by the objects named by a Henkin constant. But since the only nonlogical symbol in this language is R, this model only contains elements named by a Henkin constant (nothing else is "generated"). Hence, $\mathcal{A}$ will omit Σ by construction. Furthermore, both A and $R^{\mathcal{A}}$ will be decidable: we can simply run through the construction above, which we've seen is all decidable, until you find the sentence you're looking for (or its negation).[8] $\square$

[8] Technically, we'll have to restrict $\mathcal{A}$ back down to the language $\mathcal{L}$, but our decision procedures can still take place in the expanded language.

June 2012

1. Let A be an infinite r.e set, and let R be an r.e partial order of A. Suppose R is directed, that is, for all $a, b \in A$, there is a c such that $R(a, c) \wedge R(b, c)$. Show that there is a total recursive function f such that $\forall n\, R(f(n), f(n+1))$ and $\forall a \in A\, \exists n\, R(a, f(n))$.

> ▶ **Answer:** Enumerate the elements of A as $a_0, a_1, a_2, \ldots$. We define our function f by induction as follows. Set $f(0) = a_0$. Given that we've defined $f(n)$, enumerate the elements of R and start searching for an element b_{n+1} such that $R(a_n, b_{n+1})$ and $R(f(n), b_{n+1})$ (some such b_{n+1} is guaranteed to existed since R is directed). Set $f(n+1) = b_{n+1}$. Clearly this f is total and recursive. By construction, if $a_i \in A$, then $R(a_i, f(i+1))$. Furthermore, for all i, $R(f(i), f(i+1))$. □

2. Show that for any infinite model $\mathcal{A}$ and cardinal κ there is some elementary extension $\mathcal{B} \succcurlyeq \mathcal{A}$ whose automorphism group has size at least κ. [Hint: use Ehrenfeucht-Mostowski models.]

> ▶ **Answer:** The goal will be to show that there's a sequence $\langle X, < \rangle$ of order-indiscernibles in $\mathcal{A}$ with 2^κ-many order-preserving permutations. We can then let $\mathcal{B}$ be an Ehrenfeucht-Mostowski model built from the skolemization of $\mathsf{Th}(\mathcal{A})$ whose spine is $\langle X, < \rangle$. Each order-preserving permutation on $\langle X, < \rangle$ will induce an automorphism on $\mathcal{B}$.
>
> Consider the sequence $\langle \kappa \times \mathbb{Q}, < \rangle$ with the lexicographical ordering. Let each copy of $\mathbb{Q}$ be labeled as $\mathbb{Q}_\alpha$ for $\alpha \in \kappa$. For each $X \subseteq \kappa$, there will be a (distinct) order-preserving permutation which just shifts the elements in the copies of $\mathbb{Q}$ associated with ordinals in X by one. Since there are 2^κ many subsets of κ, there will be 2^κ-many order-preserving permutations. Hence, building an Ehrenfeucht-Mostowski model from $\kappa \times \mathbb{Q}$ will suffice. □

3. Let:

$$R_e := \{\langle a, b \rangle \in \mathbb{N}^2 \mid 2^a 3^b \in W_e\}$$
$$\mathsf{L} := \{e \in \mathbb{N} \mid R_e \text{ is a linear order on } \mathbb{N}\}$$

Show that L is Π^0_2-complete.

▶ **ANSWER:** For the sake of concreteness, I'll assume they mean strict linear order, though the solution is easily modified for non-strict linear orders. First, L is Π^0_2:

$$\begin{aligned} e \in \mathsf{L} &\Leftrightarrow R_e \text{ is a linear order on } \omega \\ &\Leftrightarrow R_e \text{ is irreflexive, asymmetric, transitive, and total} \\ &\Leftrightarrow \forall x\ \neg R_e(x,x) \wedge \forall x,y\ (R_e(x,y) \to \neg R_e(y,x)) \wedge \\ &\quad\ \forall x,y,z\ (R_e(x,y) \wedge R_e(y,z) \to R_e(x,z)) \wedge \forall x,y\ (R_e(x,y) \vee R_e(y,x)) \\ &= \Pi^0_1 \wedge \Pi^0_1 \wedge \Pi^0_2 \wedge \Pi^0_2 \\ &= \Pi^0_2 \end{aligned}$$

Next, to show Π^0_2-hardness, $\mathsf{Tot} \leqslant_m \mathsf{L}$. Define the function:

$$g(e,n) = \begin{cases} 1 & \text{if } \exists k < n\ (n = 2^k 3^{k+1} \text{ and } \phi_e(k)\downarrow) \\ \uparrow & \text{otherwise} \end{cases}$$

This function is recursive, so by s-m-n, there's a total recursive $s(e)$ such that for all x, $g(e,x) = \phi_{s(e)}(x)$. But then:

$$\begin{aligned} e \in \mathsf{Tot} &\Rightarrow \forall x\ (\phi_e(x)\downarrow) \\ &\Rightarrow \forall x\ (\exists k\ (x = 2^k 3^{k+1}) \leftrightarrow g(e,x)\downarrow) \\ &\Rightarrow \forall x\ (\exists k\ (x = 2^k 3^{k+1}) \leftrightarrow \phi_{s(e)}(x)\downarrow) \\ &\Rightarrow R_{s(e)} = \{\langle n, n+1\rangle \mid n \in \mathbb{N}\} \\ &\Rightarrow s(e) \in \mathsf{L} \\ e \notin \mathsf{Tot} &\Rightarrow \exists x\ (\phi_e(x)\uparrow) \\ &\Rightarrow \exists x,k\ (x = 2^k 3^{k+1} \wedge g(e,x)\uparrow) \\ &\Rightarrow \exists x,k\ (x = 2^k 3^{k+1} \wedge \phi_{s(e)}(x)\uparrow) \\ &\Rightarrow R_{s(e)} \text{ isn't total} \\ &\Rightarrow s(e) \notin \mathsf{L} \end{aligned}$$

(If $e \notin \mathsf{Tot}$, then $R_{s(e)}$ isn't total becuase $\langle k, k+1\rangle \notin R_{s(e)}$, and by construction, we have that for all k, $\langle k+1, k\rangle \notin R_{s(e)}$.) Hence $e \in \mathsf{Tot}$ iff $s(e) \in \mathsf{L}$, so L is Π^0_2-hard. □

4. Let $\mathcal{L}$ be a countable first-order language, let $\{P_i \mid i < \omega\}$ be distinct new predicate symbols (of any arity) and let $\mathcal{L}^n := \mathcal{L} \cup \{P_i \mid i \leqslant n\}$. Let T be a complete, consistent $\mathcal{L}^\omega$-theory, and suppose $\Sigma(x)$ is a complete 1-type of T, and let $T^n := T \cap \mathcal{L}^n$. Suppose that for each $n < \omega$, there is an $\mathcal{L}^n$-structure $\mathcal{A}_n$ such that $\mathcal{A}_n \models T^n$ and $\mathcal{A}_n$ omits $\Sigma^n(x) = \Sigma(x) \cap \mathcal{L}^n$. Show that there is an $\mathcal{L}^\omega$-structure that satisfies T but omits $\Sigma(x)$.

▶ **Answer:** It suffices to show that $\Sigma(x)$ is non-principal, for then there will be a model of T that omits it. Suppose for *reductio* that $\Sigma(x)$ is supported by $\theta(x)$. Since T is complete, and since $\Sigma(x)$ is consistent with T, $T \vdash \exists x\, \theta(x)$ and for all $\sigma \in \Sigma(x)$, $T \vdash \forall x\, (\theta(x) \to \sigma(x))$. Since proofs are finite, let's say the proof of $\exists x\, \theta(x)$ only requires $\mathcal{L}^m$. Since T is complete, T^m is also complete over $\mathcal{L}^m$, and so $\forall x\, (\theta(x) \to \sigma(x)) \in T^m$ for each $\sigma \in \Sigma^m(x)$. Hence, $\theta(x)$ supports $\Sigma^m(x)$ as well. But then $\Sigma^m(x)$ can't be omitted, $\perp$. Hence, $\Sigma(x)$ can't be supported. $\square$

5. Show that there is no partial recursive function ψ such that whenever W_e is finite, $\psi(e)\downarrow$ and $|W_e| \leqslant \psi(e)$.

▶ **Answer:** Suppose ψ is recursive. Then the following is also recursive:

$$f(e, x) = \begin{cases} 1 & \text{if } \psi(e)\downarrow \text{ and } |W_{e,x}| \leqslant \psi(e) \\ \uparrow & \text{otherwise} \end{cases}$$

Notice the following: If W_e is finite, then $f(e, x)$ is defined on cofinitely many inputs for x. If W_e is infinite, then $f(e, x)$ is defined only on finitely many inputs for x.

This function is recursive, since $W_{e,x}$ is finite, and since we know we can check $W_{e,x} \leqslant \psi(e)$ if we're given that $\psi(e)$ is defined. By *s-m-n*, $f(e, x) = \phi_{s(e)}(x)$. By the Recursion Theorem, for some d, $\phi_{s(d)} = \phi_d$.

Now consider W_d. If W_d is finite, then $\psi(d)\downarrow$ and $|W_d| \leqslant \psi(d)$. So by some stage t, $f(d, k) = \phi_{s(d)}(k) = \phi_d(k) = 1$ for all $k \geqslant t$ (i.e. is cofinite). But then W_d is infinite, $\perp$. So W_d must be infinite. But then by some stage t, $|W_{d,t}| > \psi(d)$, in which case $f(d, x) = \phi_{s(d)}(x) = \phi_d(x)$ is undefined for all $x > t$ (i.e. is finite). So then W_d is finite, $\perp$. Hence, f can't be recursive, and so ψ can't be either. $\square$

6. Let $\mathcal{L} = \{<\}$, and let T be the theory:

$$T := \{\psi \in \mathcal{L} \mid \text{for every nonempty finite linear order } \langle X, <\rangle,\ \langle X, <\rangle \models \psi\}$$

Show that T is decidable.

▶ **Answer:** **Note**: Incomplete...

Notice that the class of finite linear orders isn't first-order axiomatizable. If there were such a theory, a quick compactness argument would yield an infinite model of the theory. Thus, we can only hope to find an axiomatization of the sentences true on *all* the finite linear orders (but not on *only* the finite linear orders).

Let U be the theory of discrete linear orders with (both) endpoints. Clearly, every finite linear order satisfies U, so $U \subseteq T$. Our solution, then, has two parts. First, we must show that $T \subseteq U$. Second, we must show that U is decidable.

To get either part, we first must show that U has a nice elimination set. Let $L_n(x, y)$ ($M_n(x, y)$) be the formula stating "There are at least (at most) n-many things separating x and y (with $x < y$)" (see part (c) of this problem, page 12). Let E_n be the sentence stating "There are at least n-many things". Finally, let $\tau(x)$ state "x is a top element" and $\beta(x)$ state "x is a bottom element". Define:

$$\Sigma := \{\varphi(\bar{x}) \mid \varphi \text{ is a literal}\} \cup \{L_n(x, y), M_n(x, y), E_n \mid n \in \omega\} \cup \{\tau(x), \beta(x)\}$$

Claim (1): Σ is an elimination set for U.

▶ **PROOF:** The proof is similar to part (c) of this problem, page 12. We just need to show that we can ignore the cases where either $\tau(y)$, $\beta(y)$, or E_n appears inside the existential we're seeking to eliminate. We can clearly ignore the case where E_n appears, since this is just a sentence, and thus can be pulled outside the existential. As for the other cases, we just consider the case where $\tau(y)$ appears (the $\beta(y)$ case is symmetric).

First, notice we can ignore the case where either the b conjunct or d conjunct appears nonempty, since these are inconsistent with $\tau(y)$ (and so the existential is equivalent to $\perp$). Next, notice we can also ignore the case where the a or f conjunct appears nonempty, since these are implied by $\tau(y)$. Thus, we're left to deal with the following existential:

$$\exists y \left(\tau(y) \wedge \bigwedge_c L_{k_c}(x_c, y) \wedge \bigwedge_e M_{n_e}(x_e, y) \right)$$

But then...

CLAIM (2): If $T \vdash \varphi$, then $U \vdash \varphi$. That is, $T \subseteq U$.

▶ **PROOF:** We proceed by induction on the complexity of φ. Since φ is equivalent modulo U to a boolean combination of sentences from Σ, it suffices to show that every such boolean combination satisfies the claim.

Basis: The only non-trivial sentences from Σ are E_n. But $T \nvdash E_n$ for any $n > 1$ (the $n = 1$ case is a validity), since for any $n > 1$, there is a finite linear order of size smaller than n. Furthermore, $T \nvdash \neg E_n$ for any n. ✓

Conjunction: Suppose $\varphi := \psi \wedge \theta$, where both ψ and θ satisfy the claim. Then $T \vdash \psi \wedge \theta \Rightarrow T \vdash \psi$ and $T \vdash \theta \Rightarrow U \vdash \psi$ and $U \vdash \theta \Rightarrow U \vdash \psi \wedge \theta$. ✓

Negation:

Finally, to show U is decidable, first note that U is recursively axiomatized. Next, as shown above, U has an elimination set, and there's an effecitve way of finding the eliminating sentence for any given sentence φ. But since the only non-trivial sentences in Σ are the E_ns, it follows that φ is equivalent so some boolean combination of these E_ns. But it's straightforward to check where any such boolean combination is satisfiable or unsatisfiable: we just check to see if the constraints that boolean combination places on the size of a model is consistent. If such a boolean combination is consistent, then it's consistent with U. Hence, U has an effective decision procedure. $\square$

7. Let $C = \langle \mathbb{C}, +, \mathbb{Z} \rangle$ be a structure with the usual addition and having unary predicates for each integer. Show that every C-definable (with parameters) subset of $|C|$ is either countable or co-countable. [You may assume basic facts from linear algebra.]

8. (a) Let T be a consistent, recursively axiomatizable theory. Show that T has a model $\mathcal{A}$ such that $\mathsf{Th}(\mathcal{A})$ is Δ^0_2.

   (b) Show that there is a consistent, recursively axiomatizable T in some language $\mathcal{L}$, and a recursive set $\Sigma(x)$ of $\mathcal{L}$-formulae, such that T has models omitting $\Sigma(x)$, but whenever $\mathcal{A}$ is a model of T that omits $\Sigma(x)$, $\mathsf{Th}(\mathcal{A})$ is not arithmetic.

► **Answer (a):** Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be an enumeration of the sentences of the language. The strategy is to construct a recursive binary tree with a Δ^0_2-branch. That branch will correspond to a consistent, complete Δ^0_2 extension of T, which can then be equated to be the theory of some model of this extension.

Our tree is constructed in stages. At stage 0, we simply place T at the root of the tree. At stage s, for $s > 0$, we extend each branch with two branching nodes, one for φ_s and another for $\neg\varphi_s$. Notice that the construct of this tree is clearly recursive.

On this tree, we seek a Δ^0_2-branch as follows. First, recall that proof searches from recursively axiomatizable theories are Σ^0_1-procedures. Let B be a Σ^0_1-oracle which will tell us when a recursively axiomatiable theory Γ proves a sentence ψ (that is, $\langle \Gamma, \psi \rangle \in B$ iff $\Gamma \vdash \psi$). Our search proceeds in stages.

Stage 1: Use oracle B to determine if $T \vdash \neg\varphi_1$. If it does, add $\neg\varphi_1$ to the branch and continue the search. Otherwise, add φ_1 and continue the search. Since T is consistent, T will remain consistent if $T \nvdash \neg\varphi_1$.

Stage s: Let $\psi_1, \ldots, \psi_{s-1}$ be the sentences which have been added to the branch thus far. Since T is recursively axiomatizable, $T \cup \{\psi_1, \ldots, \psi_{s-1}\}$ is as well. So use oracle B to determine if $T \cup \{\psi_1, \ldots, \psi_{s-1}\} \vdash \neg\varphi_s$. If it does, add $\neg\varphi_s$ to the branch, and continue the search. Otherwise, add φ_s to the branch and continue the search.

This branch is a complete, consistent theory $U \supseteq T$ that was found recursively using B as an oracle. Thus, U is Δ^0_2, and so a model $\mathcal{A} \models U$ will have a Δ^0_2-theory. □

▶ **Answer (b):** Let $T = \mathsf{PA}$, and let $\Sigma(x) = \{x > \underline{n} \mid n \in \mathbb{N}\}$. Since every nonstandard model of PA is a proper end-extension of $\mathbb{N}$, it follows that the only model of T that omits $\Sigma(x)$ is the standard model. But we know $\mathsf{Th}(\mathbb{N})$ is not arithmetic. □

9. (a) Let $T \supseteq \mathsf{PA}$ be a recursively axiomatizable theory. Suppose $T \vdash \varphi$, where φ is Π^0_1. Show that if T is consistent, then φ is true.

   (b) Let φ be a Π^0_1 sentence such that $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA}) \vdash \varphi$. Show that $\mathsf{PA} \vdash \varphi$. [Hint: You may assume that the proof of the second incompleteness theory for PA can be formalized in PA. Use this to show that $\mathsf{PA} + \mathsf{Con}(\mathsf{PA}) \vdash \varphi$.]

▶ **Answer (a):** Let $T \vdash \varphi$, where φ is Π^0_1. Suppose for *reductio* that φ isn't true. So $\neg\varphi$ is true. Since $\neg\varphi$ is a true Σ^0_1, $\mathsf{PA} \vdash \neg\varphi$. But then $T \vdash \neg\varphi$, so T is inconsistent, $\bot$. □

▶ **ANSWER (b):** Suppose φ is Π^0_1 and $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA}) \vdash \varphi$, i.e. suppose that $\mathsf{PA} \vdash \neg\mathsf{Con}(\mathsf{PA}) \to \varphi$. Reasoning in PA, suppose $\mathsf{Con}(\mathsf{PA})$, and suppose for *reductio* $\neg\varphi$. Since $\neg\varphi$ is Σ^0_1, $\mathsf{Prv}_{\mathsf{PA}}(\neg\varphi)$. But since we've proven $\neg\mathsf{Con}(\mathsf{PA}) \to \varphi$, we can prove $\neg\varphi \to \mathsf{Con}(\mathsf{PA})$, and hence $\mathsf{Prv}_{\mathsf{PA}}(\neg\varphi \to \mathsf{Con}(\mathsf{PA}))$. So then $\mathsf{Prv}_{\mathsf{PA}}(\neg\varphi) \to \mathsf{Prv}_{\mathsf{PA}}(\mathsf{Con}(\mathsf{PA}))$. But then $\mathsf{Prv}_{\mathsf{PA}}(\mathsf{Con}(\mathsf{PA}))$, which can't be since we're supposing $\mathsf{Con}(\mathsf{PA})$, and by Gödel's second incompleteness theorem, we know that $\mathsf{Con}(\mathsf{PA}) \to \neg\mathsf{Prv}_{\mathsf{PA}}(\mathsf{Con}(\mathsf{PA}))$, $\bot$. So φ. Moving back outside PA, we've shown $\mathsf{PA} \vdash \mathsf{Con}(\mathsf{PA}) \to \varphi$, and so $\mathsf{PA} \vdash \varphi$. $\square$

August 2011

1. Let $\mathsf{tp}_{\mathcal{M}}(\overline{a}/X)$ denote the complete type over X with respect to $\mathcal{M}$ realized by $\overline{a} \in M$, and let $\mathsf{qftp}_{\mathcal{M}}(\overline{a}/X)$ denote the "quantifier-free" type, i.e. the type whose members are all quantifier-free. Let T be a theory. Show that the following are equivalent:

   (a) For any $\mathcal{M} \models T$, and any n-tuples $\overline{a}, \overline{b} \in M^n$, if $\mathsf{qftp}_{\mathcal{M}}(\overline{a}) = \mathsf{qftp}_{\mathcal{M}}(\overline{b})$, then $\mathsf{tp}_{\mathcal{M}}(\overline{a}) = \mathsf{tp}_{\mathcal{M}}(\overline{b})$.

   (b) T has quantifier elimination.

▶ **Answer (b ⇒ a):** Suppose for $\overline{a}, \overline{b} \in M^n$, $\mathsf{qftp}_{\mathcal{M}}(\overline{a}) = \mathsf{qftp}_{\mathcal{M}}(\overline{b})$. Let $\varphi(\overline{x}) \in \mathsf{tp}_{\mathcal{M}}(\overline{a})$. Since T has quantifier elimination, there is a quantifier-free $\psi(\overline{x})$ such that $T \vdash \forall\overline{x}\ (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$. Hence, $\psi(\overline{x}) \in \mathsf{qftp}_{\mathcal{M}}(\overline{a}) = \mathsf{qftp}_{\mathcal{M}}(\overline{b})$. Thus, $\psi(\overline{x}) \in \mathsf{tp}_{\mathcal{M}}(\overline{b})$, so $\varphi(\overline{x}) \in \mathsf{tp}_{\mathcal{M}}(\overline{b})$. The other direction is symmetric. □

<u>**Note**</u>: The other direction, as stated, is more difficult, and I'm not sure if it's correct. There are two ways to fix the problem statement so that it's more manageable. One is to assume T is complete. Another is to assume that (a) can be rephrased to say $\mathsf{qftp}_{\mathcal{M}}(\overline{a}) = \mathsf{qftp}_{\mathcal{N}}(\overline{b}) \Rightarrow \mathsf{tp}_{\mathcal{M}}(\overline{a}) = \mathsf{tp}_{\mathcal{N}}(\overline{b})$, where $\mathcal{M}, \mathcal{N} \models T$ may be different. We deal with each modification below.

▶ **Answer (a ⇒ b):** **Assuming T is complete**: Let $\varphi(\overline{x})$ be a formula, and let $\overline{c}, \overline{d}$ be new constants added to the language. Then by (a), $T \cup \left\{\psi(\overline{c}) \leftrightarrow \psi(\overline{d}) \,\middle|\, \psi \text{ is quantifier-free}\right\} \vdash \varphi(\overline{c}) \leftrightarrow \varphi(\overline{d})$. So by Compactness, there's some finite number of these formulae, say $\psi_1, \ldots, \psi_n$, such that $T \cup \left\{\psi_1(\overline{c}) \leftrightarrow \psi_1(\overline{d}), \ldots, \psi_n(\overline{c}) \leftrightarrow \psi_n(\overline{d})\right\} \vdash \varphi(\overline{c}) \leftrightarrow \varphi(\overline{d})$. Call σ an *admissibility condition* if it is a formula of the form $\bigwedge_i \psi_i(\overline{x}) \wedge \bigwedge_j \neg\psi_j(\overline{x})$ for $i, j \leqslant n$ with $i \neq j$, and let $\Sigma(\overline{x})$ be the set of all admissibility conditions consistent with $\varphi(\overline{x})$. If $\Sigma(\overline{x}) = \varnothing$, then $T \vdash \forall\overline{x}\ \neg\varphi(\overline{x})$, so $\varphi(\overline{x})$ is just equivalent to $\perp$. Otherwise, since $|\Sigma(\overline{x})|$ is finite, $\bigvee_{i\in\Sigma} \sigma_i(\overline{c})$ is a well-formed quantifier-free sentence. We claim:

Claim: $T \vdash \varphi(\overline{c}) \leftrightarrow \bigvee_{i\in\Sigma} \sigma_i(\overline{c})$.

▶ **Proof:** Clearly $T \vdash \varphi(\bar{c}) \to \bigvee_{i \in \Sigma} \sigma_i(\bar{c})$, since those exhaust the possibilities consistent with $\varphi(\bar{c})$. As for the converse, suppose $T \nvdash \bigvee_{i \in \Sigma} \sigma_i(\bar{c}) \to \varphi(\bar{c})$. That means for at least one disjunct, $\sigma_k(\bar{c})$, $T \nvdash \sigma_k(\bar{c}) \to \varphi(\bar{c})$. Since T is complete, $T \vdash \sigma_k(\bar{c}) \wedge \neg\varphi(\bar{c})$, and thus $T \vdash \exists \bar{x}\ (\sigma_k(\bar{x}) \wedge \neg\varphi(\bar{x}))$. But since $\sigma_k(\bar{x}) \in \Sigma(\bar{x})$, some model of T must satisfy $\sigma_k(\bar{x}) \wedge \varphi(\bar{x})$. Let $\mathcal{A}$ be such a model, with say $\mathcal{A} \models \sigma_k(\bar{a}) \wedge \varphi(\bar{a})$ for $\bar{a} \in A$. Since $T \vdash \exists \bar{x}\ (\sigma_k(\bar{x}) \wedge \neg\varphi(\bar{x}))$, there must be some $\bar{b} \in A$ such that $\mathcal{A} \models \sigma_k(\bar{b}) \wedge \neg\varphi(\bar{b})$. But $\mathcal{A} \models \sigma_k(\bar{a}) \wedge \sigma_k(\bar{b})$, so $\bar{a}$ and $\bar{b}$ agree on each ψ_i from above, and hence by assumption they must agree on φ, $\perp$. □

Since T doesn't mention $\bar{c}$, $T \vdash \forall \bar{x}\ (\varphi(\bar{x}) \leftrightarrow \bigvee_{i \in \Sigma} \sigma_i(\bar{x}))$. □

▶ **Answer (a ⇒ b): Assuming (a) is restated as, "For any $\mathcal{A}, \mathcal{B} \models T$ and any $\bar{a} \in A$, $\bar{b} \in B$, $\mathsf{qftp}_{\mathcal{A}}(\bar{a}) = \mathsf{qftp}_{\mathcal{B}}(\bar{b}) \Rightarrow \mathsf{tp}_{\mathcal{A}}(\bar{a}) = \mathsf{tp}_{\mathcal{B}}(\bar{b})$.":** Let $\Sigma(\bar{x}) := \{\psi(\bar{x}) \mid T \vdash \forall \bar{x}\ (\varphi(\bar{x}) \to \psi(\bar{x}))$ and ψ is quantifier-free$\}$. Let $\bar{c}$ be a set of new constants.

Claim: $T \cup \Sigma(\bar{c}) \vdash \varphi(\bar{c})$

▶ **Proof:** Suppose not. Let $\mathcal{A} \models T \cup \Sigma(\bar{c}) \cup \{\neg\varphi(\bar{c})\}$.

Claim: $T \cup \mathsf{qftp}_{\mathcal{A}}(\bar{c}) \cup \{\varphi(\bar{c})\}$ is satisfiable.

▶ **Subproof:** Suppose not. Then for some $\theta(\bar{x}) \in \mathsf{qftp}_{\mathcal{A}}(\bar{c})$, $T \vdash \theta(\bar{c}) \to \neg\varphi(\bar{c})$, and hence $T \vdash \forall \bar{x}\ (\varphi(\bar{x}) \to \neg\theta(\bar{x}))$. So $\neg\theta(\bar{x}) \in \Sigma(\bar{x})$, and thus $\mathcal{A} \models \neg\theta(\bar{c})$. But then it follows that $\neg\theta(\bar{x}) \in \mathsf{qftp}_{\mathcal{A}}(\bar{c})$, $\perp$. □

Let $\mathcal{B} \models T \cup \mathsf{qftp}_{\mathcal{A}}(\bar{c}) \cup \{\varphi(\bar{c})\}$. Since $\mathcal{B} \models \mathsf{qftp}_{\mathcal{A}}(\bar{c})$, we have $\mathsf{qftp}_{\mathcal{A}}(\bar{c}^{\mathcal{A}}) = \mathsf{qftp}_{\mathcal{B}}(\bar{c}^{\mathcal{B}})$. So by (a), $\mathsf{tp}_{\mathcal{A}}(\bar{c}^{\mathcal{A}}) = \mathsf{tp}_{\mathcal{B}}(\bar{c}^{\mathcal{B}})$, which can't be, since $\mathcal{A} \models \neg\varphi(\bar{c})$, but $\mathcal{B} \models \varphi(\bar{c})$, $\perp$. □

Hence, for some $\theta(\bar{x}) \in \Sigma(\bar{x})$, $T \vdash \forall \bar{x}\ (\theta(\bar{x}) \to \varphi(\bar{x}))$. □

2. Prove that if a countable theory T is $\aleph_0$-categorical, then all of T's models are $\aleph_0$-saturated.

▶ **Answer:** Let $\mathcal{A} \models T$. It suffices to show that for any finitely many $\overline{a} \in A$, and for any 1-type $p(x)$ over $\overline{a}$, $p(x)$ is realized in $\mathcal{A}$. Since T is $\aleph_0$-categorical, by the Engeler-Ryll-Nardzewski-Svenonius Theorem, all of T's n-types (without parameters) must be principal.

Claim: $p(x)$ is principal.

▶ **Proof:** Define the following n-type without parameters:

$$q(x, \overline{y}) := \{\varphi(x, \overline{y}) \mid \varphi(x, \overline{a}) \in p(x)\}$$

Since q is an n-type without parameters, there's a formula $\theta(x, \overline{y})$ such that $T \vdash \forall x, \overline{y} \ (\theta(x, \overline{y}) \to \varphi(x, \overline{y}))$ for all $\varphi \in q(x, \overline{y})$. But then $T \vdash \forall x \ (\theta(x, \overline{a}) \to \varphi(x, \overline{a}))$ for all $\varphi(x, \overline{a}) \in p(x)$, and hence $\theta(x, \overline{a})$ supports $p(x)$. □

Now, by Vaught's Test, T is complete, and hence $p(x)$ cannot be omitted from any model of T. Thus, $\mathcal{A}$ realizes $p(x)$. □

3. Prove that there is no saturated model of the theory of dense linear orders without endpoints of size $\aleph_\omega$.

▶ **Answer:** Suppose $\mathcal{A}$ was a saturated model of DLO of size $\aleph_\omega$. Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ be models of DLO such that for each $i \in \omega$, $|A_i| = \aleph_i$ and $A = \bigcup_i A_i$ (which we can do, since $\aleph_\omega$ is irregular). Since $\mathcal{A}$ is saturated, it must realize the type $p_i(x) = \{x > a \mid a \in A_i\}$ for each $i \in \omega$, say by a_i. Let $p(x) = \{x > a_i \mid i \in \omega\}$. By compactness, $p(x)$ is satisfiable, so $\mathcal{A}$ must also realize $p(x)$. But it can't, since $A = \bigcup_i A_i$. $\perp$ □

4. Let $M = {}^\omega 2$ be the set of countable binary sequences, and let $\mathcal{M}$ be the structure with carrier set M in the language $\mathcal{L} = \{E_n \mid n \in \omega\}$ where E_n is a binary relation symbol to be interpreted so that if $a, b \in M$, $E_n(a, b)$ iff for all $i \leqslant n$, $a_i = b_i$ (where s_i denotes the i^{th} element in the sequence s).

(a) Present a list of axioms for $\mathsf{Th}(\mathcal{M})$, and prove that your theory is complete.

(b) Describe all the 1-types realizes in $\mathcal{M}$.

(c) Describe all the 2-types realizes in $\mathcal{M}$.

5. Prove the following:

(a) If T is an infinite, finitely branching recursive tree on $\mathbb{N}$, then there is an infinite branch f of T that is recursive in $0''$.

(b) There is a finitely branching recursive tree on $\mathbb{N}$ such that whenever f is an infinite branch of T, $0'$ is recursive in f.

▶ **Answer (a):** Let T be an infinite, finitely branching recursive tree. By König's Lemma, there is an infinite branch in this tree. We'll perform a Δ^0_3-search for an infinite branch f in stages as follows. Suppose we're at stage s of our search, having found $f_s := f \restriction s$. Enumerate the nodes $m_1, \ldots, m_n$ immediately extending f_s (which can be done since f_s is finite, and T is recursive). Using Inf as an oracle, search through the m_is for a node with infinitely-many nodes extending it. Extend f_s to include the first such m_i you find. This will result in an infinite branch that was found in a Δ^0_3-way. □

▶ **Answer (b):** Let A and B be two Δ^0_2-inseparable Σ^0_2 sets (see this problem, page 5). We build a binary tree T such that any Δ^0_2 branch would separate A and B, which implies that there cannot be such a branch.

We construct our tree along Σ^0_2-constructions for sets A and B. At stage s of the construction of our tree, we will extend every branch f of height s by nodes 0 and 1 so long as:

(i) for all $t < s$, $t \in A_s \Rightarrow f(t) = 1$

(ii) for all $t < s$, $t \in B_s \Rightarrow f(t) = 0$

If one of these conditions isn't met, then don't extend the branch any further.

At the end of the construction, we end up with an infinite binary tree, so there will be an infinite branch. However, if f is an infinite branch, then at every stage s of the construction, if $f(s) = 1$, then f was extended because either $s \in A$, or because $s \in \overline{A \cup B}$. Similarly, if $f(s) = 0$, then f was extended because either $s \in B$ or $s \in \overline{A \cup B}$. The result, then, is that $f = \chi_C$ for some $C \supseteq A$ where $C \cap B = \varnothing$. Hence, if f was constructed in a Δ^0_2-way, C would Δ^0_2-separate A and B, $\perp$. □

6. Show that there are $a, b \in \mathbb{N}$ such that $\forall x \ (\phi_a(x) = b \wedge \phi_b(x) = a)$.

▶ **ANSWER:** Let $f(x, y, z) = x$. By *s-m-n*, there's a recursive $s(x, y)$ such that $f(x, y, z) = \phi_{s(x,y)}(z)$. Since s is recursive, by the Uniform Recursion Theorem, there's a recursive $t(x)$ such that $\phi_{s(x,t(x))} = \boxed{\phi_{t(x)} = x}$.

Now let $g(x, y, z) = y$, and let $h(x, z) = g(x, t(x), z)$. By *s-m-n*, there's a recursive $r(x)$ such that $h(x, z) = \phi_{r(x)}(z) = t(x)$. By the Recursion Theorem, there's an a such that $\phi_{r(a)} = \boxed{\phi_a = t(a)}$. Let $\boxed{b = t(a)}$. Then by the boxed equations, $\phi_b = \phi_{t(a)} = a$ and $\phi_a = t(a) = b$. □

7. For $A \subseteq \mathbb{N}^2$, let $A_n = \{y \mid \langle n, y \rangle \in A\}$. Show that $P := \{e \mid \forall n \ (W^2_e)_n \text{ is finite}\}$ is complete at some level of the hierarchy.

▶ **ANSWER:** First, we'll show P is Π^0_3. Let $t(e, n)$ be the recursive function such that $W_{t(e,n)} = (W^2_e)_n$ (it's easy to check by *s-m-n* this is recursive). Then:

$$\begin{aligned} P(e) &\Leftrightarrow \forall n \ ((W^2_e)_n \text{ is finite}) \\ &\Leftrightarrow \forall n \ (W_{t(e,n)} \text{ is finite}) \\ &\Leftrightarrow \forall n \ \mathsf{Fin}(t(e, n)) \\ &= \Pi^0_3 \end{aligned}$$

Next, we'll show that P is Π^0_3-hard by showing that $\overline{P}$ is Σ^0_3-hard. Let A be a Σ^0_3 set. Then, since Inf is Π^0_2-complete, there's a recursive $g(e, x)$ such that:

$$A(e) \Leftrightarrow \exists n \ \mathsf{Inf}(g(e, n))$$

So define the function:

$$f(e,n,x) :\simeq \begin{cases} 1 & \text{if } \phi_{g(e,n)}(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

This is recursive, so by *s-m-n*, there's a recursive s such that $f(e,n,x) \simeq \phi_{s(e)}(n,x)$. But then:

$$\begin{aligned}
e \in A &\Rightarrow \exists n \ \mathsf{Inf}(g(e,n)) \\
&\Rightarrow \exists n \ \left(\{x \mid \phi_{g(e,n)}(x)\downarrow\} \text{ is infinite}\right) \\
&\Rightarrow \exists n \ \left(\{x \mid f(e,n,x)\downarrow\} \text{ is infinite}\right) \\
&\Rightarrow \exists n \ \left(\{x \mid \phi_{s(e)}(n,x)\downarrow\} \text{ is infinite}\right) \\
&\Rightarrow \exists n \ \left(\left(W^2_{s(e)}\right)_n \text{ is infinite}\right) \\
&\Rightarrow s(e) \in \overline{P} \\
e \notin A &\Rightarrow \forall n \ \mathsf{Fin}(g(e,n)) \\
&\Rightarrow \forall n \ \left(\{x \mid \phi_{g(e,n)}(x)\downarrow\} \text{ is finite}\right) \\
&\Rightarrow \forall n \ \left(\{x \mid f(e,n,x)\downarrow\} \text{ is finite}\right) \\
&\Rightarrow \forall n \ \left(\{x \mid \phi_{s(e)}(n,x)\downarrow\} \text{ is finite}\right) \\
&\Rightarrow \forall n \ \left(\left(W^2_{s(e)}\right)_n \text{ is finite}\right) \\
&\Rightarrow s(e) \in P
\end{aligned}$$

This completes the reduction. □

8. Let $\mathcal{L}$ be the language of PA, and let Val be the set of $\mathcal{L}$-validities. Show that Val is Σ^0_1-complete. [You may assume that every recursive relation is representable in Q.]

▶ **Answer:** Val is clearly Σ^0_1, since we can recursively enumerate proofs in first-order logic generally. To show that Val is Σ^0_1-hard, we'll show that $K \leqslant_m \mathsf{Val}$. Since K is Σ^0_1, it is weakly represented in Q by some formula, $\varphi(x)$. But Q is finitely axiomatizable, so where Q_{ax} are the finitely many axioms of Q, $\mathsf{Q} \vdash \theta$ iff $\vdash \bigwedge \mathsf{Q}_{ax} \to \theta$, i.e. iff $\ulcorner \bigwedge \mathsf{Q}_{ax} \to \theta \urcorner \in \mathsf{Val}$. Hence, we have that $n \in K$ iff $\ulcorner \bigwedge \mathsf{Q}_{ax} \to \varphi(\underline{n}) \urcorner \in \mathsf{Val}$, and since coding sentences into gödel numbers can be done in a primitive recursive way, this complete the reduction. $\square$

9. Let $T := \mathsf{PA} \cup \{\mathsf{Prv}_{\mathsf{PA}}(\ulcorner\theta\urcorner) \to \theta \mid \theta \text{ is a sentence}\}$. Let $S := \mathsf{PA} + \mathsf{Con}(\mathsf{PA})$. Show that $T \nvdash \mathsf{Con}(S) \to \mathsf{Con}(T)$.

▶ **Answer:** We call the schema $\mathsf{Prv}_{\mathsf{PA}}(\ulcorner\theta\urcorner) \to \theta$ the **soundness schema**. We argue for two claims:

Claim (1): T is consistent.

▶ **Proof (1):** If T is inconsistent, then for some $\psi_1, \dots, \psi_n \in T$, we'd have that $\mathsf{PA} \vdash \bigvee_{i=1}^{n} (\mathsf{Prv}_{\mathsf{PA}}(\ulcorner\psi_i\urcorner) \wedge \neg\psi_i)$. By distributivity, $\mathsf{PA} \vdash \bigvee_{i=1}^{n} \mathsf{Prv}_{\mathsf{PA}}(\ulcorner\psi_i\urcorner)$. Since this is Σ_1^0, and since PA is Σ_1^0-sound, $\bigvee_{i=1}^{n} \mathsf{Prv}_{\mathsf{PA}}(\ulcorner\psi_i\urcorner)$ must be true. Hence, $\mathsf{PA} \vdash \psi_i$ for some i. But since $\mathsf{PA} \vdash \bigvee_{i=1}^{n} (\mathsf{Prv}_{\mathsf{PA}}(\ulcorner\psi_i\urcorner) \wedge \neg\psi_i)$, for at least one such ψ_i, we must have $\mathsf{PA} \vdash \neg\psi_i$, $\perp$. Hence, T is consistent. $\square$

Claim (2): $T \vdash \mathsf{Con}(S)$.

▶ **Proof (2):** First note that $\mathsf{Prv}_{\mathsf{PA}}(\ulcorner\perp\urcorner) \to \perp$ is an instance of the soundness schema, and so $T \vdash \mathsf{Con}(\mathsf{PA})$. Now, reasoning in T, suppose $\neg\mathsf{Con}(\mathsf{PA} + \mathsf{Con}(\mathsf{PA}))$. Then $\mathsf{PA} \vdash \neg\mathsf{Con}(\mathsf{PA})$, i.e. $\mathsf{Prv}_{\mathsf{PA}}(\ulcorner\neg\mathsf{Con}(\mathsf{PA})\urcorner)$. But then, by the soundness schema, $\neg\mathsf{Con}(\mathsf{PA})$, and we already know $\mathsf{Con}(\mathsf{PA})$, $\perp$. Hence, $T \vdash \mathsf{Con}(\mathsf{PA} + \mathsf{Con}(\mathsf{PA}))$, i.e. $T \vdash \mathsf{Con}(S)$. $\square$

Hence, if it were the case that $T \vdash \mathsf{Con}(S) \to \mathsf{Con}(T)$, then we'd have $T \vdash \mathsf{Con}(T)$. So by Gödel's second incompleteness theorem, T would be inconsistent, $\perp$. $\square$

January 2011

1. Suppose E is a r.e equivalence relation with finitely many equivalence classes. Show that E is recursive.

▶ **Answer:** See part (a) of this problem, page 35. □

2. Prove that there is a r.e set having the property that its complement is infinite but the set itself meets every infinite r.e set non-trivially (i.e. it's not disjoint with any infinite r.e set; i.e. its complement has no infinite r.e subset).

▶ **Answer:** We construct such a set A in stages as follows. Set $A_0 = \varnothing$. At stage $s > 0$, search through each of $W_{0,s}, W_{1,s}, \dots, W_{s,s}$ and determine whether any of $W_{i,s} \cap A_{s-1} = \varnothing$. If there isn't any, set $A_s = A_{s-1}$. If there are some, search through the least such $W_{i,s}$ for an $n \geqslant 2i$. If there is such an n in that $W_{i,s}$, set $A_s = A_{s-1} \cup \{n\}$; otherwise, go to the next $W_{j,s}$. Repeat this process until you find such an n or run out of sets to check, in which case, just set $A_s = A_{s-1}$.

At any stage, s, $|A_s| \leqslant s$, and $A_s \subseteq \{0, \dots, 2s\}$; hence $|\overline{A_s}| \geqslant |A_s|$. But for any given infinite set W_e, there will always be a stage at which an element $n \geqslant 2e$ is added to W_e; and eventually, it will be the least such W_i with this property. Hence, A_s is infinite, and $A_s \cap W_e \neq \varnothing$ for some s, if W_e is infinite. □

3. Show that for any infinite model $\mathcal{A}$, there is a proper elementary extension $\mathcal{B} \succ \mathcal{A}$ with an elementary embedding $f : \mathcal{B} \to \mathcal{B}$ such that $A = \bigcap_{n=1}^{\infty} f^n(B)$. [Hint: Ehrenfeucht-Mostowski]

Note: For most solutions we tried, the idea was to build an Ehrenfeucht-Mostowski model from $\mathcal{A}$, with indiscernibles $c_1, c_2, c_3, \dots$, and then have f send $c_i \mapsto c_{i+1}$. Then eventually $\bigcap_{n=1}^{\infty} f^n(B)$ wouldn't contain any of our original indiscernibles. The problem with this approach, however, is that there might be some element d which was added to the EM model that could be written in an infinite number of ways as a term of our original order-indiscernibles. If this were so, we could have this element appearing in $f^n(B)$ for arbitrarily large n, and hence in the intersection of all $f^n(B)$. This solution (credited to Alex Kruckman) gets around this worry.

▶ **ANSWER:** Let $\mathcal{M} \succcurlyeq \mathcal{A}$ be $|A|^+$-saturated. We say a type $p(\overline{x})$ is **finitely satisfiable** in A if for every $\varphi(\overline{x}) \in p(\overline{x})$, there's a $\overline{a} \in A$ such that $\mathcal{A} \models \varphi(\overline{a})$.

CLAIM (1): There is a type $p(x) \in \mathsf{St}_1(M)$ such that $p(x)$ is finitely satisfiable in $\mathcal{A}$ but is not realized in $\mathcal{M}$.

▶ **PROOF (1):** Add a new constant c to the language, and let:

$$p(c) := \{\neg\varphi(c,\overline{a}) \mid \overline{a} \in M \text{ and } \mathcal{A} \models \neg\exists x,\overline{y}\,\varphi(x,\overline{y})\} \cup \{c \neq m \mid m \in M\}$$

Consider $\neg\varphi_1(c,\overline{a}),\ldots,\neg\varphi_n(c,\overline{a})$ and $c \neq m_1,\ldots,c \neq m_k$. Since $\mathcal{A} \models \neg\exists x,\overline{y}\,\varphi(x,\overline{y})$, and since only finitely many m_is are mentioned, we can always find another element $b \in A$ such that $c^{\mathcal{M}} = b$ will satisfy each of these sentences. Thus, $p(c)$ is satisfiable by compactness, i.e. $p(x) \in \mathsf{St}_1(M)$. And clearly while it's finitely satisfiable in $\mathcal{A}$, it's not realized in $\mathcal{M}$. □

So take a $p(x)$ over M that's finitely satisfiable in $\mathcal{A}$ but not realized in $\mathcal{M}$. For notational convenience, let $p \upharpoonright X$ denote the set of formulae in φ that only mention elements from X as parameters.

CLAIM (2): p is invariant over A, i.e. for all $\overline{c},\overline{d} \in M$ and $\overline{a} \in A$, if $\mathsf{tp}_{\mathcal{M}}(\overline{c}/A) = \mathsf{tp}_{\mathcal{M}}(\overline{d}/A)$, then $\varphi(x,\overline{c},\overline{a}) \in p(x) \Leftrightarrow \varphi(x,\overline{d},\overline{a}) \in p(x)$.

▶ **PROOF (2):** Suppose $\varphi(x,\overline{c},\overline{a}) \in p(x)$ but $\varphi(x,\overline{d},\overline{a}) \notin p(x)$. Since p is complete, that means $\neg\varphi(x,\overline{d},\overline{a}) \in p(x)$. Since, $p(x)$ is finitely satisfiable in A, and since $\varphi(x,\overline{c},\overline{a}) \wedge \neg\varphi(x,\overline{d},\overline{a}) \in p(x)$, there's a $b \in A$ such that $\mathcal{M} \models \varphi(b,\overline{c},\overline{a}) \wedge \neg\varphi(b,\overline{d},\overline{a})$. But $b \in A$, so $\varphi(b,\overline{x},\overline{a}) \in \mathsf{tp}_{\mathcal{M}}(\overline{c}/A)$ and yet $\neg\varphi(b,\overline{x},\overline{a}) \in \mathsf{tp}_{\mathcal{M}}(\overline{d}/A)$. Hence $\mathsf{tp}_{\mathcal{M}}(\overline{c}/A) \neq \mathsf{tp}_{\mathcal{M}}(\overline{d}/A)$. The other direction is similar. □

Add countably many constants $b_0,b_1,b_2,\ldots$ to the language, and add to $\mathsf{Th}(\mathcal{M})$ sentences $p(b_0) \upharpoonright A$ and for each $i \in \omega$ $p(b_{i+1}) \upharpoonright A \cup \{b_0,\ldots,b_i\}$. Since $\mathcal{M}$ is $|A|^+$-saturated, all of these types are realized in $\mathcal{M}$.

Claim (3): $\{b_i \mid i \in \omega\}$ is a set of order-indiscernibles over A.

▶ **Proof (3):** By induction on the length of the sequence.

Basis: If $\mathcal{M} \models \varphi(b_i, \overline{a})$ with $\overline{a} \in A$, then $\varphi(x, \overline{a}) \in p \restriction A$, and hence $\varphi(x, \overline{a}) \in p \restriction A \cup \{b_0, \ldots, b_{j-1}\}$. So $\mathcal{M} \models \varphi(b_j, \overline{a})$. ✓

Induction: Suppose we already have that where $i_1 < \cdots < i_{n+1}$ and $j_1 < \cdots < j_{n+1}$, $\mathsf{tp}_{\mathcal{M}}(b_{i_1}, \ldots, b_{i_n}/A) = \mathsf{tp}_{\mathcal{M}}(b_{j_1}, \ldots, b_{j_n}/A)$. Allow me to write $\overline{b}_i$ for the n-many $b_{i_1} < \cdots < b_{i_n}$, and similarly for $\overline{b}_j$. Then by Claim (2), we have that for any formula $\varphi(x, \overline{y}, \overline{a})$, $\varphi(x, \overline{b}_i, \overline{a}) \in p(x) \Leftrightarrow \varphi(x, \overline{b}_j, \overline{a}) \in p(x)$. So taking a formula $\varphi(x, \overline{y}, \overline{a})$, we have:

$$\begin{aligned}
\varphi(x, \overline{y}, \overline{a}) \in \mathsf{tp}_{\mathcal{M}}\left(\overline{b}_i, b_{i+1}/A\right) &\Leftrightarrow \varphi(x, \overline{b}_i, \overline{a}) \in \mathsf{tp}_{\mathcal{M}}\left(b_{i+1}/A, \overline{b}_i\right) \\
&\Leftrightarrow \varphi(x, \overline{b}_i, \overline{a}) \in p(x) \\
&\Leftrightarrow \varphi(x, \overline{b}_j, \overline{a}) \in p(x) \\
&\Leftrightarrow \varphi(x, \overline{b}_j, \overline{a}) \in \mathsf{tp}_{\mathcal{M}}\left(b_{j+1}/A, \overline{b}_j\right) \\
&\Leftrightarrow \varphi(x, \overline{y}, \overline{a}) \in \mathsf{tp}_{\mathcal{M}}\left(\overline{b}_j, b_{j+1}/A\right)
\end{aligned}$$

This completes the proof. □

Now, let $\mathcal{B} := \langle A \cup \{b_i \mid i \in \omega\}\rangle_{\mathcal{M}}$, and let $f : B \to B$ send $b_i \mapsto b_{i+1}$ and keep A fixed. Clearly, $A \subseteq \bigcap_{n=1}^{\infty} f^n(B)$. Now, let $c \in \bigcap_{n=1}^{\infty} f^n(B)$. Then $c \in B$, so there must be a term t_1 such that $c = t_1(\overline{b}_i)$, where $b_{i_1} < \cdots < b_{i_n}$. But $c \in f^{i_n+1}(B)$, so there must be some term t_2 such that $c = t_2(\overline{b}_j)$, where $b_{i_n} < b_{j_1} < \cdots < b_{j_m}$. Hence, the formula (where m is the length of $\overline{x}$), $t_2(\overline{x}) = t_1(\overline{b}_i) \in \mathsf{tp}_{\mathcal{M}}\left(\overline{b}_j/A, \overline{b}_i\right)$. Now since by construction $\mathcal{M} \models p \restriction (A \cup \{b_0, \ldots, b_{j_m-1}\})(b_{j_m})$, we have that $t_1(b_1, \ldots, b_{m-1}, x_m) = t_2(\overline{b}_i) \in p(x)$. But since p is finitely satisfiable in A, there must be some $a_m \in A$ such that $\mathcal{M} \models t_1(b_{j_1}, \ldots, b_{j_{m-1}}, a_m) = t_2(\overline{b}_i)$. Hence, $t_1(x_1, \ldots, x_{m-1}, a_m) = t_2(\overline{b}_i) \in \mathsf{tp}_{\mathcal{M}}\left(b_{j_1}, \ldots, b_{j_{m-1}}/A, \overline{b}_i\right)$. Continuing in this way, we can get a sequence $a_1, \ldots, a_m \in A$ realizing $t_1(\overline{x}) = t_2(\overline{b}_i)$. But then $c = t_1(\overline{a}) \in A$. □

4. Let $\mathcal{L} = \langle 0, + \rangle$. Prove that the extension $\langle \mathbb{Q}, 0, + \rangle \subseteq \langle \mathbb{R}, 0, + \rangle$ is elementary.

▶ **Answer:** By Tarski-Vaught, it suffices to show that for every $\varphi(\overline{x}, y)$, and for any $\overline{a} \in \mathbb{Q}$, if $\mathbb{R} \models \exists y\, \varphi(\overline{a}, y)$, then for some $b \in \mathbb{Q}$, $\mathbb{R} \models \varphi(\overline{a}, b)$. We will proceed by first showing that $\langle \mathbb{R}, 0, + \rangle$ has quantifier elimination. Then we'll take $\varphi(\overline{a}, y)$, replace it with its quantifier-free equivalent $\psi(\overline{a}, y)$, and then prove the claim by brute force.

Claim: $\langle \mathbb{R}, 0, + \rangle$ has quantifier-elimination.

▶ **Proof:** It suffices to check that, where $\theta(\overline{x}, y)$ is a conjunction of literals, there's a quantifier-free $\psi(\overline{x})$ such that for all $\overline{a} \in \mathbb{R}$, $\mathbb{R} \models (\exists y\, \theta(\overline{a}, y) \leftrightarrow \psi(\overline{a}))$. The literals of this language are equivalent to formulae of the form "$y = \sum_{i=1}^{n} c_i x_i$" or "$y \neq \sum_{i=1}^{n} c_i x_i$" (where '$c_i x_i$' is just an abbreviation for '$x_i + \cdots + x_i$' with c_i-many summands).[a] If $\theta(\overline{x}, y)$ contains any of the former kind, then we can take one instance of "$\sum_{i=1}^{n} c_i a_i$" and replace y with it everywhere in $\theta(\overline{a}, y)$. So it suffices to check just the cases where $\theta(\overline{x}, y)$ is a conjunction of literals of the form "$y \neq \sum_{i=1}^{n} c_i x_i$". But in this case, since there are only finitely many inequalties, there will always be such a y in $\mathbb{R}$, so we can simply replace $\theta(\overline{a}, y)$ with $\top$. □

[a] I guess it should be more general than this, e.g. $dy = \sum_{i=1}^{n} c_i x_i$, but there's ways of reducing those cases to cases where the literals are just of this form (e.g. common multiples).

Now, take $\mathbb{R} \models \exists y\, \varphi(\overline{a}, y)$, with $\overline{a} \in \mathbb{Q}$, where we've ensured φ is quantifier-free by quantifier-elimination. Thus, it suffices to consider the case where $\varphi(\overline{a}, y)$ is of the form:

$$\bigwedge_{k=1}^{n} y = \sum_{i_k=1}^{n_k} c_{i_k} a_{i_k} \wedge \bigwedge_{l=1}^{n} y \neq \sum_{i_l=1}^{n_l} c_{i_l} a_{i_l}$$

If there are any conjuncts in the big conjunction on the left, then whatever realizes this formula in $\mathbb{R}$ must be a rational number, since it is the sum of finitely many rationals. On the other hand, if there aren't any conjuncts in the big conjunction on the left, then there will always be a rational number that realizes these inequalities. Either way, there is a $b \in \mathbb{Q}$ such that $\mathbb{R} \models \varphi(\overline{a}, b)$. □

5. Show that there is no total function f for which $f \leqslant_T \varnothing'$ and for every e, if W_e is finite, then $f(e) = |W_e|$.

> ▶ **ANSWER:** See this problem, page 50. The "$\leqslant$" in that problem can be changed to "$=$" without much change in the proof. For a slightly more difficult problem, see this problem, page 40. □

6. Let $\mathcal{L}$ be a countable first-order language, let $\mathcal{L}' = \mathcal{L} \cup \{P_i \mid i < \omega\}$ for some new unary predicates $\{P_i \mid i < \omega\}$, and let T' be a complete, consistent $\mathcal{L}'$-theory. Suppose $\Sigma(x)$ is a set of $\mathcal{L}$ formulae such that for each $n \in \omega$, $T' \cap (\mathcal{L} \cup \{P_i \mid i < n\})$ has a model omitting $\Sigma(x)$. Prove that T' itself has a model omitting $\Sigma(x)$.

> ▶ **ANSWER:** See this problem, page 49. □

7. Consider the semiring:

$$\mathbb{Z}[x]_{\geqslant 0} = \{f(x) \in \mathbb{Z}[x] \mid \exists N \in \mathbb{N}\ \forall n > N\ \ (f(n) \geqslant 0)\}$$

Order $\mathbb{Z}[x]_{\geqslant 0}$ by:

$$f \leqslant g \Leftrightarrow \exists h\ \in \mathbb{Z}[x]_{\geqslant 0}\, f + h = g$$

Prove or disprove: $\langle \mathbb{Z}[x]_{\geqslant 0}, \leqslant, +, \times, 0, 1 \rangle \models \mathsf{PA}$.

> ▶ **ANSWER:** Define the sentence:
>
> $$\varphi := \forall x\ \left[\exists y\ (x = \underline{2} \cdot y) \vee \exists y\ (x = (\underline{2} \cdot y) + 1)\right]$$
>
> We know that $\mathsf{PA} \vdash \varphi$. Consider, however, $f(x) = x$. $f \in \mathbb{Z}[x]_{\geqslant 0}$, since 0 can witness N. But there is no $g \in \mathbb{Z}[x]_{\geqslant 0}$ such that $f = 2 \cdot g$, since g must have coefficients from $\mathbb{Z}$. And there is no $g \in \mathbb{Z}[x]_{\geqslant 0}$ such that $f = 2 \cdot g + 1$, since otherwise $x - 1 = 2 \cdot g$, which would again violate the fact that g must have coefficients from $\mathbb{Z}$. Hence, f is a counter-example to φ in $\mathbb{Z}[x]_{\geqslant 0}$, so $\mathbb{Z}[x]_{\geqslant 0} \not\models \mathsf{PA}$. □

8. Show by example (and prove that your example has the requisite properties) that there is a complete theory T in a countable language for which there is exactly one 1-type relative to T but continuum many 2-types.

June 2010

1. Suppose $\varphi(x, y)$ and $\psi(y)$ are two $\mathcal{L}_{\mathsf{PA}}$-formulae. Let:

$$\Sigma(x) := \mathsf{Th}(\mathbb{N}) \cup \{\varphi(x, \underline{n}) \mid \mathbb{N} \models \psi(\underline{n})\} \cup \{\neg\varphi(x, \underline{n}) \mid \mathbb{N} \models \neg\psi(\underline{n})\}$$

be a consistent (partial) 1-type. Prove that if $M \supset \mathbb{N}$, and $\mathcal{M} \equiv \mathbb{N}$, then there is a $b \in M$ such that $\mathcal{M} \models \Sigma(b)$.

▶ **Answer:** Let $\mathcal{M} \models \mathsf{Th}(\mathbb{N})$ be nonstandard. Consider the formula:

$$\theta(x) := \exists y\ \forall z < x\ \ (\varphi(y, z) \leftrightarrow \psi(z))$$

That is, $\theta(\underline{n})$ roughly says, "There is a witness to the first n φ-formulae of $\Sigma(x)$" (where by the k^{th} φ-formula, I mean $\varphi(x, \underline{k})$ if $\mathbb{N} \models \psi(\underline{k})$ and $\neg\varphi(x, \underline{k})$ otherwise).

If there were an $m \in \mathbb{N}$ such that $\mathbb{N} \not\models \theta(\underline{m})$, that would mean that there is no witness to the first m φ-formulae of $\Sigma(x)$, i.e. some finite subset of $\Sigma(x)$ would be unsatisfiable, $\perp$. Hence, $\mathbb{N} \models \theta(\underline{m})$ for all $m \in \mathbb{N}$, and so $\mathbb{N} \models \forall x\ \theta(x)$. Since $\mathcal{M} \equiv \mathbb{N}$, $\mathcal{M} \models \forall x\ \theta(x)$. But then for any nonstandard element $b \in M$, $\mathcal{M} \models \theta(b)$, i.e. $\mathcal{M}$ has a witness to all of the φ-formulae in $\Sigma(x)$. □

2. Show that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every arithmetically definable set can be computed from any function g for which $\forall n\ \ (g(n) \geqslant f(n))$.

3. Prove that $\mathsf{Th}(\mathbb{Z}, <)$ is decidable.

▶ **Answer:** See this problem, page 12, part (a) for a proof that $\mathsf{Th}(\mathbb{Z}, <)$ is finitely axiomatizable and complete. From these two facts, it follows that $\mathsf{Th}(\mathbb{Z}, <)$ is decidable. □

4. Prove the following:

   (a) If $\langle T, \leqslant\rangle$ is an infinite, finitely branching tree, then there is an infinite subset $S \subseteq T$ which is linearly ordered by $\leqslant$.

   (b) There is a recursively finitely branching tree for which there is no infinite recursive $S \subseteq T$ linearly ordered by $\leqslant$.

► **Answer (a, b):** See this problem, page 59 for both parts. To say that $S \subseteq T$ is linearly ordered just means that S is a branch of T. Part (a) is just König's lemma, which is the proof given in part (a), except with no concern regarding the complexity. Part (b) can be shown using Δ^0_1-inseparable r.e sets instead. □

5. We say that T **eliminates** $\exists^\infty$ if for each formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, there is a formula $\theta(\overline{z})$ such that for all $\mathcal{M} \models T$, $\mathcal{M} \models \theta(\overline{b}) \Leftrightarrow \left\{\overline{a} \in M^n \mid \mathcal{M} \models \varphi(\overline{a}, \overline{b})\right\}$ is infinite.

   (a) Show that T eliminates $\exists^\infty$ iff for each $\varphi(\overline{x}, \overline{y})$, there is a number n_φ such that for any model $\mathcal{M} \models T$ and any $\overline{b} \in M^m$, if the set $\left\{\overline{a} \in M^n \mid \mathcal{M} \models \varphi(\overline{a}, \overline{b})\right\}$ is finite, it has size at most n_φ.

   (b) Show that if T is countable and T does not eliminate $\exists^\infty$, then there is an uncountable model $\mathcal{M} \models T$ and a countably infinite definable (with parameters) set $X \subseteq M$.

► **Answer (a):**

($\Leftarrow$) Let $\varphi(\overline{x}, \overline{y})$ be a formula, and let n_φ be as above. Define:

$$\theta(\overline{y}) := \exists \overline{x}_0, \ldots, \overline{x}_{n_\varphi} \left(\bigwedge_{\substack{i,j \leqslant n_\varphi \\ i \neq j}} (\overline{x}_i \neq \overline{x}_j) \wedge \bigwedge_{i=0}^{n_\varphi} \varphi(\overline{x}_i, \overline{y}) \right)$$

where $\overline{x}_i \neq \overline{x}_j$ abbreviates $\bigvee_{k=1}^{n} x_{i,k} \neq x_{j,k}$ (i.e. the tuple $\overline{x}_i$ is not completely identical to the tuple $\overline{x}_j$). Clearly, for any given $\overline{b}$, if there are infinitely many $\overline{a}$ such that $\mathcal{M} \models \varphi(\overline{a}, \overline{b})$, then $\mathcal{M} \models \theta(\overline{b})$. Conversely, if $\mathcal{M} \models \theta(\overline{b})$, then the size of $\left\{\overline{a} \in M^n \mid \mathcal{M} \models \varphi(\overline{a}, \overline{b})\right\}$ is at least $n_\varphi + 1$, so it must be infinite, by the definition of n_φ. □

($\Rightarrow$) Suppose T eliminates $\exists^\infty$, and let $\varphi(\overline{x}, \overline{y})$ be a formula, where $\theta(\overline{y})$ is the eliminating formula. Suppose for *reductio* that there is no such number n_φ. That means for each $k \in \omega$, there is a model $\mathcal{M}_k \models T$ and a $\overline{b}_k \in M_k$ such that $\left\{\overline{a} \in M_k^n \mid \mathcal{M}_k \models \varphi(\overline{a}, \overline{b}_k)\right\}$ is finite, but has size greater than k. Now, add some new constants $\overline{c}$ to the language, and define:

$$\Gamma := T \cup \{\neg\theta(\overline{c})\} \cup \{\exists_{\geqslant n}\overline{x}\ \varphi(\overline{x},\overline{c}) \mid n \in \omega\}$$

Take any finite $\Gamma_0 \subseteq \Gamma$. It will only contain finitely many instances of $\exists_{\geqslant n}\overline{x}\ \varphi(\overline{x},\overline{c})$. So pick the greatest k such that we have $\exists_{\geqslant k}\overline{x}\ \varphi(\overline{x},\overline{c}) \in \Gamma_0$. By hypothesis, there is a $\overline{b}_k \in M_k$ such that $\{\overline{a} \in M_k^n \mid \mathcal{M}_k \models \varphi(\overline{a},\overline{b}_k)\}$ is finite but greater than k. Hence, setting $c_i^{\mathcal{M}_k} = b_i$ will satisfy Γ_0. By Compactness, Γ is satisfiable. But then for some $\mathcal{N} \models T$, $\mathcal{N} \models \neg\theta(\overline{c})$, while the set $\{\overline{a} \in N^n \mid \mathcal{N} \models \varphi(\overline{a},\overline{c})\}$ is infinite, contrary to the definition of θ, $\perp$. □

6. Prove that if $X \subseteq \mathbb{N}$ is an infinite r.e set, then there is a recursive $f : X \to X$ such that f has no fixed points, but $f \circ f = \mathtt{id}_X$.

▶ **Answer:** Let $a_0, a_1, a_2, \ldots$ be a recursive enumeration (without repetitions) of X. Define f as follows:

$$f(a_i) = \begin{cases} a_{i+1} & \text{if } i \text{ is even} \\ a_{i-1} & \text{if } i \text{ is odd} \end{cases}$$

That is, f is a sequence of loops of length 2 between elements of X. Then clearly $f(f(a_i)) = a_i$. Furthermore, the graph of f is r.e., since we can just list the members of f as we list the members of X. Hence, f is recursive. □

7. Give an example (with proof that your example works) of a universal model which is not saturated.

▶ **Answer:** Given our analysis of this problem, page 12, we can use the model $\langle \mathbb{Q}^- \times \mathbb{Z}, < \rangle$, where $\mathbb{Q}^-$ is the nonpositive half of $\mathbb{Q}$. We know that $\mathbb{Q} \times \mathbb{Z}$ can be embedding into $\mathbb{Q}^- \times \mathbb{Z}$, since $\mathbb{Q}$ can be embedded into any proper initial segment of itself. But $\langle \mathbb{Q}^- \times \mathbb{Z}, < \rangle$ isn't saturated, since it omits the type saying, "x is infinitely far ahead of $\langle 0, 0 \rangle$". □

8. Does there exist a consistent, recursive $T \supseteq \mathsf{PA}$ for which $T \vdash \neg\mathsf{Con}(T)$? Justify your answer.

▶ **Answer:** Surprisingly, yes! Let $T = \mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$. By Gödel's second incompleteness theorem, T is consistent; and clearly T is recursive, since PA is. Furthermore, $T \vdash \neg\mathsf{Con}(\mathsf{PA})$, and $\mathsf{PA} \vdash \neg\mathsf{Con}(\mathsf{PA}) \to \neg\mathsf{Con}(\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA}))$ (since PA knows that, if it's inconsistent, so are all of its extensions). Hence, $T \vdash \neg\mathsf{Con}(T)$.[9] $\square$

[9] An important point about this problem is that this is made possible by the fact that T is ω-inconsistent. Suppose T were ω-consistent. Then T would be Σ^0_1-sound. But then, if $T \vdash \neg\mathsf{Con}(T)$, $\neg\mathsf{Con}(T)$ would have to be true, $\perp$. Hence, this situation can only arise if T is ω-inconsistent. In that case, the idea is that while T *thinks* it's inconsistent, it's actually wrong.

January 2010

1. Prove or disprove: There is a partial recursive $f(x)$ such that whenever W_e is finite, $f(e)\downarrow$ and $|W_d| \leqslant f(e)$.

> ▶ **ANSWER:** See this problem, page 50. □

2. Let $\mathcal{L} = \{f\}$, where f is a unary function symbol. Prove that the empty $\mathcal{L}$-theory has a model companion. (And I quote: "In principle, this could be solved by abstract nonsense, but we would prefer to see an axiomatization of the model companion and then a proof that your axiomatization works.")

3. Let $\mathcal{M} \models \mathsf{PA}$ and let $\varphi(x, y)$ be an $\mathcal{L}_{\mathsf{PA}}$-formula. Let $c \in M$ with $c > 0$, and let $S \subseteq \{a \in M \mid a < c\}$. Suppose that for all $a \in S$, $\mathcal{M} \models \exists x\ \varphi(x, a)$. Show that there is a $b \in M$ such that for all $a \in S$, $\mathcal{M} \models \exists x < b\ \varphi(x, a)$.

> ▶ **ANSWER:** Define the formula (where a and b are treated as variables for readability):
>
> $$\theta(u) := \exists b\ \forall a < u\ \ (\exists x\ \varphi(x, a) \to \exists x < b\ \varphi(x, a))$$
>
> We know that for all $a \in S$, $\mathcal{M} \models \exists x\ \varphi(x, a)$,[a] so if $\mathcal{M} \models \theta(c)$, then $\mathcal{M} \models \exists b\ \forall a < u\ \exists x < b\ \varphi(x, a)$. Thus, there's a b such that for all $a \in S$, $\mathcal{M} \models \exists x < b\ \varphi(x, a)$. So we just need to show that $\mathcal{M} \models \theta(c)$.
>
> In fact, we'll show that $\mathcal{M} \models \forall u\ \theta(u)$. First, note that $\mathsf{PA} \vdash \theta(0)$, since the outermost bounded universal becomes trivial. Next, we want to show that $\mathsf{PA} \vdash \forall u\ \ (\theta(u) \to \theta(u + 1))$. Reasoning in PA, suppose for *reductio* that $\theta(u)$ is true, but not $\theta(u + 1)$. That means that for all b, there's an $a < u + 1$ such that $\varphi(x, a)$ has a witness, but not one less than b. Let these witnesses be $d_0, \ldots, d_v$ (where $v \leqslant u$). Consider $d := \max(d_0, \ldots, d_v) + 1$. By hypothesis, there must be some $a < u + 1$ which has a witness to $\varphi(x, a)$, but not one less than d. But regardless of the a we pick, if $\varphi(x, a)$ has a witness, then $d_a < d$ is a witness, $\perp$. Hence, moving outside of PA, we conclude $\mathsf{PA} \vdash \forall u\ \ (\theta(u) \to \theta(u + 1))$.
>
> So by the induction schema, $\mathsf{PA} \vdash \forall u\ \theta(u)$, from which it follows that $\mathcal{M} \models \theta(c)$. □
>
> [a] It's okay if there's such a where $a \notin S$.

4. Let $E := \{e \mid W_e = \varnothing\}$. Prove or disprove: $\mathsf{Inf} \leqslant_T E$.

► **Answer:** See this problem, page 81, for a proof that Inf is Π^0_2-complete. As for E:

$$\begin{aligned} E(e) &\Leftrightarrow W_e = \varnothing \\ &\Leftrightarrow \forall x\ (x \notin W_e) \\ &= \Pi^0_1 \end{aligned}$$

That is, E is at most Π^0_1. So $\mathsf{Inf} \nleqslant_T E$. □

5. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and for each $i \in \omega$, let $\mathcal{A}_i \preccurlyeq \mathcal{M}$. Let $\mathcal{L}' = \mathcal{L} \cup \{P_i \mid i \in \omega\}$, where P_i is a new unary predicate symbol, and let $\mathcal{M}'$ be the expansion of $\mathcal{M}$ to $\mathcal{L}'$ via the interpretation $P_i^{\mathcal{M}'} = A_i$. Assume that for each finite $F \subseteq \omega$, $\bigcap_{i \in F} \mathcal{A}_i \preccurlyeq \mathcal{M}$. Show that there is an $\mathcal{N}' \succcurlyeq \mathcal{M}'$ for which there is a $\mathcal{B} \preccurlyeq \mathcal{N}$, where $\mathcal{N} := \mathcal{N}' \restriction \mathcal{L}$, such that:

(i) for each $i \in \omega$, $B \subseteq P_i^{\mathcal{N}'}$

(ii) $B \cap M = \bigcap_{i \in \omega} A_i$

► **Answer:** Add a new unary predicate B to the language, and consider the following theory:

$$\begin{aligned} T := {} & \mathsf{ElDiag}\,(\mathcal{M}) \cup \{\forall x\ (B(x) \to P_i(x)) \mid i \in \omega\} \\ & \cup \left\{ \neg B(a) \,\middle|\, a \in M - \bigcap_{i \in \omega} A_i \right\} \cup \left\{ B(a) \,\middle|\, a \in \bigcap_{i \in \omega} A_i \right\} \\ & \cup \left\{ \forall \bar{x} \left[\left(\bigwedge_{k=1}^{n} B(x_k) \wedge \exists y\ \varphi(\bar{x}, y) \right) \to \exists y\ (B(y) \wedge \varphi(\bar{x}, y)) \right] \,\middle|\, \varphi \text{ is an } \mathcal{L}\text{-formula} \right\} \end{aligned}$$

Suppose T is satisfiable. Then there's an $\mathcal{N}' \succcurlyeq \mathcal{M}$ satisfying T. By Tarski-Vaught, $\mathcal{B} := \langle B^{\mathcal{N}'} \rangle_{\mathcal{N}'} \preccurlyeq \mathcal{N}'$. Also, $B_i^{\mathcal{N}'} \subseteq P_i^{\mathcal{N}'}$. Now, if $a \in B \cap M$, since $B \cap (M - \bigcap_{i \in \omega} A_i) = \varnothing$, $a \in \bigcap_{i \in \omega} A_i$, i.e. $B \cap M \subseteq \bigcap_{i \in \omega} A_i$. And if $a \in \bigcap_{i \in \omega} A_i$, then clearly $a \in M$ and $a \in B^{\mathcal{N}'}$. Hence, it suffices to show that T is finitely satisfiable.

If $\Gamma \subseteq T$ is a finite number of sentences not in $\mathsf{ElDiag}(\mathcal{M})$, then it only mentions finitely-many P_i's, say with $i \in F$ for some finite F. Hence, if we take $B^{\mathcal{N}} = \bigcap_{i \in F} A_i$, which by hypothesis gives us that $\langle B^{\mathcal{M}} \rangle_{\mathcal{M}} \preccurlyeq \mathcal{M}$, then M expanded in this way will model Γ. □

6. For the sake of this problem, you may assume that every finite partial order can be extended to a linear order.

   (a) Show that, if $\langle A, R \rangle$ is a partial order, then there is a linear order $<$ on A such that $< \supseteq R$.

   (b) Show that, if $\langle \omega, R \rangle$ is a recursive partial order, then there is a Δ^0_2 set $S \subseteq \omega^2$ such that $R \subseteq S$ and $\langle \omega, S \rangle$ is a linear order.

▶ **Answer (a):** Let $\mathcal{L} = \{R, <\}$, and define the $\mathcal{L}$-theory T to be the theory containing $\mathsf{ElDiag}(\mathcal{A})$ which says that $<$ is a linear order and that $R \subseteq <$. Since any finite partial order can be extended to a linear order, every finite subset of T will be consistent. So by Compactness, T will be satisfiable, say by $\mathcal{B} \models T$. But then $\langle A, R^{\mathcal{B}} \restriction A, <^{\mathcal{B}} \restriction A \rangle$ will be a linear order extending A. □

7. Give an example of a model $\mathcal{M}$ (you choose the language) such that there are elements $a, b \in M$, a model $\mathcal{N} \succcurlyeq \mathcal{M}$, and an automorphism $\sigma : N \to N$ such that $\sigma(a) = b$, but there is no automorphism $\tau : M \to M$ for which $\tau(a) = b$.

▶ **Answer:** See this problem, page 21. Part (b) gives several examples of a model in which there is an element $a \in M$ that's not definable, and yet every automorphism of M fixes a. But part (a) shows that, if a is not definable, we can always elementarily extend to another model $\mathcal{N}$ in which there is an automorphism that doesn't fix a. □

8. Let $\mathcal{M} \models \mathsf{PA}$.

   (a) Show that there is no $\varphi(x, y)$ such that for every definable set $D \subseteq M$, there is some parameter $b \in M$ for which $D = \{a \in M \mid \mathcal{M} \models \varphi(a, b)\}$.

(b) Show that for any $c \in M$, there is a formula $\theta(x,y)$ such that for any definable $D \subseteq \{a \in M \mid 0 \leqslant a < c\}$, there is a parameter $d \in M$ where $D = \{a \in M \mid \mathcal{M} \models \theta(a,d)\}$.

▶ **Answer (a):** Consider the set $C := \{a \in M \mid \mathcal{M} \models \neg\varphi(a,a)\}$. C is clearly a definable set (by formula $\neg\varphi(x,x)$), so by hypothesis, there's a $b \in M$ such that $C = \{a \in M \mid \mathcal{M} \models \varphi(a,b)\}$. But $b \in C \Leftrightarrow \mathcal{M} \models \varphi(b,b)$ $\Leftrightarrow \mathcal{M} \not\models \neg\varphi(b,b) \Leftrightarrow b \notin C$, $\perp$. □

▶ **Answer (b):** The idea is that this parameter will be a code for D. Let $c \in M$ be fixed, and define $\theta(x,y) := p_x \mid y$ (where p_i is the i^{th} prime). Let $D < c$ be defined by $\varphi(x)$. Suppose we show that:

$$\mathsf{PA} \vdash \forall z\, \exists y\, \forall x < z\ \ (\varphi(x) \leftrightarrow \theta(x,y))$$

Then we have $\mathcal{M} \models \exists y\, \forall x < c\ \ (\varphi(x) \leftrightarrow \theta(x,y))$. Hence, for some $d \in M$, $\mathcal{M} \models \forall x < c\ \ (\varphi(x) \leftrightarrow \theta(x,d))$. But if $b \geqslant c$, then $\mathcal{M} \models \neg\varphi(b)$ and (by appropriately picking our d) $\mathcal{M} \models \neg\theta(b,d)$. Hence, actually, we have $\mathcal{M} \models \forall x\ \ (\varphi(x) \leftrightarrow \theta(x,d))$. Thus, $D = \{a \in M \mid \mathcal{M} \models \theta(a,d)\}$. So it suffices to show:

Claim: $\mathsf{PA} \vdash \forall z\, \exists y\, \forall x < z\ \ (\varphi(x) \leftrightarrow \theta(x,y))$.

▶ **Proof:** By induction. Clearly, $\mathsf{PA} \vdash \exists y\, \forall x < 0\ \ (\varphi(x) \leftrightarrow \theta(x,y))$. Next, reasoning in PA, suppose $\exists y\ \forall x < n\ \ (\varphi(x) \leftrightarrow \theta(x,y))$. Either n satisfies φ or it doesn't. If it does, then taking the witness for $\forall x < n\ \ (\varphi(x) \leftrightarrow \theta(x,y))$, we can extend it by multiplying it by p_n to obtain a witness for $\forall x < n+1\ \ (\varphi(x) \leftrightarrow \theta(x,y))$. If it doesn't, then our original witness will automatically give us as a witness to $\forall x < n+1\ \ (\varphi(x) \leftrightarrow \theta(x,y))$. Either way, $\exists y\, \forall x < n+1\ \ (\varphi(x) \leftrightarrow \theta(x,y))$. □

This completes the proof. □

August 2009

1. Prove or disprove the following statements:

   (a) There exists an e such that $W_e = \{x \mid \phi_e(x)\uparrow\}$.

   (b) There exists an e such that $W_e = \{x \mid \phi_x(e)\downarrow\}$.

▶ **Answer (a):** False. By definition, $W_e = \mathsf{dom}(\phi_e) = \{x \mid \phi_e(x)\downarrow\}$. □

▶ **Answer (b):** True. Define:

$$f(e,x) = \begin{cases} 1 & \text{if } \phi_x(e)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

This function is recursive, so by s-m-n, there's a total recursive $s(x)$ such that $f(e,x) = \phi_{s(e)}(x)$, and thus $W_{s(e)} = \{x \mid \phi_x(e)\downarrow\}$. By the Recursion Theorem, there's an a such that $\phi_{s(a)} = \phi_a$. Thus, $W_a = \{x \mid \phi_x(a)\downarrow\}$. □

2. Show that every countable structure in a countable language has an $\aleph_0$-homogeneous elementary extension.

▶ **Answer:** Let $\mathcal{A}$ be a countable structure. We will build an elementary chain of models $\mathcal{A} \preccurlyeq \mathcal{B}_1 \preccurlyeq \mathcal{B}_2 \preccurlyeq \mathcal{B}_3 \preccurlyeq \cdots$ such that $|B_i| = \aleph_0$ for each i, and such that if $\mathsf{tp}_{\mathcal{B}_i}(\overline{a}) = \mathsf{tp}_{\mathcal{B}_i}(\overline{b})$, then for each $c \in \mathcal{B}_i$, there's a $d \in \mathcal{B}_{i+1}$ such that $\mathsf{tp}_{\mathcal{B}_{i+1}}(\overline{a}, c) = \mathsf{tp}_{\mathcal{B}_{i+1}}(\overline{b}, d)$. It will follow that $\mathcal{B} \coloneqq \bigcup_i \mathcal{B}_i$ will be a $\aleph_0$-homogeneous countable elementary extension of $\mathcal{A}$.

Suppose we've built $\mathcal{B}_i$ with the desired properties. First, list all the tuples $\langle \overline{a}_i, \overline{b}_i, c_i \rangle \in B_i^{2n+1}$ such that $\mathsf{tp}_{\mathcal{B}_i}(\overline{a}) = \mathsf{tp}_{\mathcal{B}_i}(\overline{b})$. We will build an "inner" elementary chain of models $\mathcal{B}_{i,j}$ as follows. First, set $\mathcal{B}_{i,0} \coloneqq \mathcal{B}_i$. Next, given $\mathcal{B}_{i,j}$, since $\mathsf{tp}_{\mathcal{B}_{i,j}}(\overline{a}_j) = \mathsf{tp}_{\mathcal{B}_{i,j}}(\overline{b}_j)$, and since $\mathsf{tp}_{\mathcal{B}_{i,j}}(\overline{a}_j, c_j)$ is a consistent type, there is an elementary extension $\mathcal{B}_{i,j+1} \succcurlyeq \mathcal{B}_{i,j}$ (which we can ensure is countable by Downward Löwenheim-Skolem) such that for some $d_j \in \mathcal{B}_{i,j+1}$, $\mathsf{tp}_{\mathcal{B}_{i,j+1}}(\overline{a}_j, c_j) = \mathsf{tp}_{\mathcal{B}_{i,j+1}}(\overline{b}_j, d_j)$. Finally, take $\mathcal{B}_{i+1} \coloneqq \bigcup_j \mathcal{B}_{i,j}$. Then clearly $\mathcal{B}_{i+1}$ has the desired properties. □

3. If A, B are sets of natural numbers, define the **symmetric difference operation** as $A \triangle B := (A - B) \cup (B - A)$. Say $A \approx B$, that is A is **almost** B, if $A \triangle B$ is finite. Finally, define the set $P := \{\langle x, y\rangle \mid W_x \approx W_y\}$. Show that P is Σ^0_3-complete. [Hint: Show that Cof is Σ^0_3-complete first.]

▶ **ANSWER:** Suppose Cof is Σ^0_3-complete. We'll show that P is Σ^0_3, and that Cof $\leqslant_m P$. First:

$$\begin{aligned} P(a,b) &\Leftrightarrow W_a \approx W_b \\ &\Leftrightarrow (W_a - W_b) \cup (W_b - W_a) \text{ is finite} \\ &\Leftrightarrow \exists s, n \ [\mathsf{Seq}\,(s) \wedge \mathsf{lh}\,(s) = n \wedge \forall i, j < n\ ((s)_i \neq (s)_j) \wedge \\ &\qquad\qquad \forall x\ (x \in W_a \wedge x \notin W_b \leftrightarrow \exists i < n\ ((s)_i = x))] \\ &\quad \wedge \textit{vice versa} \text{ for } a \text{ and } b \\ &= \exists s, n\ (\Delta^0_1 \wedge \Delta^0_1 \wedge \Delta^0_1 \wedge \forall x\ (\Pi^0_1 \wedge \Sigma^0_1 \leftrightarrow \Delta^0_1)) \\ &= \Sigma^0_3 \end{aligned}$$

Next, let $W_e = \mathbb{N}$ for some fixed e. Let $f(x) = \langle x, e\rangle$. Clearly f is recursive, and $x \in$ Cof iff $f(x) \in P$. Hence, so long as we can show Cof is Σ^0_3-complete, we'll have a proof that P is as well.

The proof of Cof being Σ^0_3-complete is rather involved. It involves another priority-esque argument... But showing Cof is Σ^0_3 is at least straightforward: in defining Cof(e), just replace "$x \in W_a \wedge x \notin W_b$" with "$x \notin W_e$". This will still be Σ^0_3, as is easy to check. □

4. Let T be the theory of discrete linear orders without endpoints.

   (a) Describe the prime model of T, and justify your answer.

   (b) Describe the countably saturated model of T, and justify your answer.

   (c) Give an example of a countably homogeneous model of T that's neither prime nor saturated. (FALSE)

▶ **ANSWER (a,b):** See this problem, page 12. There, part (a) proves background needed for parts (b) and (c). □

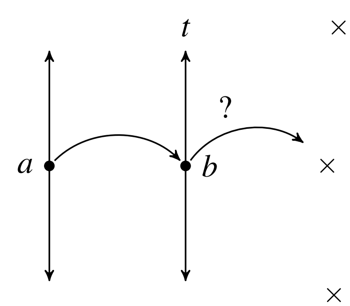
▶ **Answer (c):** Not possible. Suppose $\mathcal{A} \models T$ is homogeneous, and that $\mathcal{A} \ncong \langle \mathbb{Z}, < \rangle$. Thus, for some nontrivial linear order L, $\mathcal{A} \cong \langle L \times \mathbb{Z}, < \rangle$. We will show that $L \cong \mathbb{Q}$, from which it will follow that $\mathcal{A}$ is saturated. To show this, since $\mathbb{Q}$ is the only countable dense linear order without endpoints, it suffices to show that L is dense and without endpoints.

Before proceeding, recall from part (c) of this problem, page 12 that T has an elimination set. This elimination set only contains formulae with two free variables; hence, every formula with one free variable must be equivalent to some boolean combination of these formulae where the first and second variables are the same. But (by induction) every such boolean combination is equivalent modulo T to either $\top$ or $\perp$. Hence, every element in A satisfies the same exact formulae with one free variable, viz. the formulae which are equivalent to $\top$.

Claim (1): L doesn't have a top element.

▶ **Proof (1):** Suppose for *reductio* that $t \in L$ is a top element. Let $a, b \in A$ be such that a's first coordinate isn't t, but b's first coordinate is. By the above remarks, the map $f : \{a\} \to A$ such that $a \mapsto b$ is partial elementary. And since $\mathcal{A}$ is homogeneous, it follows there's an automorphism $\sigma \supseteq f$.

Since σ is an automorphism, it preserves order, and hence it must send b to some element above b (since $a < b$). But a, b satisfy the 2-type saying "x is infinitely far behind y"; thus, so must $\sigma(a), \sigma(b)$. Hence, $\sigma(b)$ must be infinitely far ahead of b. But this is impossible since b lies on the last $\mathbb{Z}$-chain in $\mathcal{A}$, $\perp$. $\square$

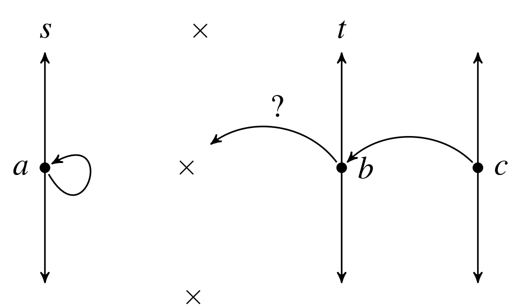


A similar proof can be used to show L has no bottom element.

Claim (2): L is dense.

▶ **Proof (2):** Suppose for *reductio* that there are two elements $s, t \in L$ with $s < t$ such that there is no $r \in L$ where $s < r$ and $r < t$. Let $a, b, c \in A$ be elements such that a's first coordinate is s, b's is t, and c's is some element above t (which by Claim (1) we know must be possible). Let $f : \{a, c\} \to A$ map $a \mapsto a$ and $c \mapsto b$. By part (c) of this problem, 12, a, c satisfy the same 2-type as a, b, viz. the one which says "x is infinitely far below y". Hence, f is partial elementary, and since $\mathcal{A}$ is homogeneous, there is an automorphism $\sigma \supseteq f$.

Since σ is an automorphism, it must preserve order, and hence it must send b to some element between a and b. $\sigma(b)$ can't be on the same $\mathbb{Z}$-chain as $\sigma(c) = b$, since otherwise $\sigma(b), \sigma(c)$ would not satisfy the type saying "x is infinitely far below y", whereas b, c would. And $\sigma(b)$ can't be on the same $\mathbb{Z}$-chain as $\sigma(a) = a$, since $\sigma(a), \sigma(b)$ would also not satisfy that type, whereas a, b would. Hence, $\sigma(b)$ must lie on a $\mathbb{Z}$-chain between a and b. But there is no such $\mathbb{Z}$-chain, since there is no element between s and t, $\perp$. □



Hence, $L \cong \mathbb{Q}$. □

5. Let $\mathcal{M} \models \mathsf{PA}$ be nonstandard, and let $X \subseteq \mathbb{N}$. Show that if there is an $a \in M$ which codes X, then for all nonstandard b, there is a $c < b$ that codes X.

> ▶ **Answer:** If X is finite, then some finite sequence s will code X, and hence for all nonstandard b, $s < b$, in which case the claim is immediate. So suppose X is infinite, i.e. it's coded by some nonstandard a. Consider the formula (with parameter a):
>
> $$\varphi(x, a) := \mathsf{Seq}(x) \wedge \exists n \ (\mathsf{lh}(x) = n \wedge \forall m < n \ (p_m \mid x \leftrightarrow p_m \mid a))$$
>
> where p_k is the k^{th} prime, which is something that can be expressed in the language of PA. Then this will be true for arbitrarily large x: for any $k \in \mathbb{N}$ such that $\mathcal{M} \models \varphi(k, a)$, there will always be a $k' \in \mathbb{N}$ such that $\mathcal{M} \models k' > k \wedge \varphi(k', a)$, just by considering the sequence which includes the next prime p_n such that $n \in X$. Hence, by overspill, there must be arbitrarily small c such that $\mathcal{M} \models \varphi(c, a)$, i.e. for all nonstandard b, there is a $c < b$ such that c codes X. □

6. Let T be a countable theory with infinite models. Show that there is an uncountable model $\mathcal{M} \models T$ such that, up to isomorphism, there are only countably many finitely generated substructures of $\mathcal{M}$.

7. Show that there exists a set A recursive in $0'$ which is not a boolean combination of r.e sets.

> ▶ **Answer:** Just as we can enumerate the sentences of a propositional language, so too we can enumerate the boolean combinations of r.e sets W_e. Let these boolean combinations be enumerated by $B_0, B_1, B_2, \ldots$. Clearly, each B_i is recursive in $0'$. But now define the set $C := \{e \mid e \notin B_e\}$ (akin to K). Since each B_e is recursive in $0'$, so is C. But if $C = B_d$, then $B_d(d)$ iff $C(d)$ iff $\neg B_d(d)$, $\perp$. □

8. Show that $\mathsf{Th}(\mathbb{Z}, 0, +)$ is decidable.

January 2009

1. Prove or disprove: If $\mathcal{A} \prec \mathcal{B}$, and $a \in B - A$, then a is not definable.

> ▶ **Answer:** True. Suppose a were definable with $\varphi(x)$. That is, $\mathcal{B} \models \forall x\ (x = a \leftrightarrow \varphi(x))$. Then $\mathcal{B} \models \exists! x\ \varphi(x)$. However, since $\mathcal{A} \preccurlyeq \mathcal{B}$, $\mathcal{A} \equiv \mathcal{B}$, so $\mathcal{A} \models \exists! x\ \varphi(x)$. But then there's an element $a' \in A$ such that $\mathcal{A} \models \varphi(a')$. Again, since $\mathcal{A} \preccurlyeq \mathcal{B}$, $\mathcal{B} \models \varphi(a')$. But $a \neq a'$, since $a' \in A$ and $a \notin A$. Hence $\mathcal{B} \not\models \forall x\ (\varphi(x) \to x = a)$, $\perp$. □

2. Show that Inf is Π^0_2-complete.

> ▶ **Answer:** First, to show Inf is Π^0_2:
>
> $$\begin{aligned} \mathsf{Inf}(e) &\Leftrightarrow W_e \text{ is infinite} \\ &\Leftrightarrow \forall n\ \exists s\ [\mathsf{Seq}\,(s) \wedge \mathsf{lh}\,(s) > n \wedge \forall i,j \leqslant n (i \neq j \to (s)_i \neq (s)_j) \wedge \\ &\qquad\qquad \forall i \leqslant n\ ((s)_i \in W_e)] \\ &= \forall n\ \exists s\ (\Delta^0_1 \wedge \Delta^0_1 \wedge \Delta^0_1 \wedge \Sigma^0_1) \\ &= \Pi^0_2 \end{aligned}$$
>
> To show that it's Π^0_2-hard, let A be Π^0_2 so that $A(e)$ iff $\forall x\ \exists y\ R(x,y,e)$ where R is Δ^0_1. Define the function:
>
> $$f(e,n) = \begin{cases} 1 & \text{if } \forall x < n\ \exists y\ R(x,y,e) \\ \uparrow & \text{otherwise} \end{cases}$$
>
> This function is recursive, so by s-m-n, there's a total recursive s such

that $f(e,n) = \phi_{s(e)}(n)$. But then:

$$\begin{aligned}
e \in A &\Rightarrow \forall x\, \exists y\, R(x,y,e) \\
&\Rightarrow \forall x\, (f(e,x)\downarrow) \\
&\Rightarrow \forall x\, (\phi_{s(e)}(x)\downarrow) \\
&\Rightarrow s(e) \in \mathsf{Inf} \\
e \notin A &\Rightarrow \exists x\, \forall y\, \neg R(x,y,e) \\
&\Rightarrow \forall z > x\, (f(e,z)\uparrow) \\
&\Rightarrow |W_{s(e)}| < \aleph_0 \\
&\Rightarrow s(e) \notin \mathsf{Inf}
\end{aligned}$$

This completes the reduction. □

3. (a) Does every nonstandard model of PA have a proper elementary substructure?

   (b) Does every nonstandard model of PA have a proper Σ^0_1-elementary substructure?

▶ **Answer (a):** No. Let $\mathcal{M} \models \mathsf{PA}$ be nonstandard. Consider the set $X := \{a \in M \mid a \text{ is definable without parameters}\}$, and let $\mathcal{A} := \langle X \rangle_{\mathcal{M}}$. Clearly $\mathcal{A} \subseteq \mathcal{M}$. To show that it's elementary, suppose $\mathcal{M} \models \exists x\, \varphi(x, \overline{a})$, for some parameters $\overline{a} \in X$. Then since $\mathcal{M} \models \mathsf{PA}$, and since for any $\psi(\overline{x}, y)$, $\mathsf{PA} \vdash \forall \overline{x}\, (\exists y\, \psi(\overline{x}, y) \to \exists y\, (\psi(\overline{x}, y) \wedge \forall z < y\, \neg\psi(\overline{x}, z)))$, we have that $\mathcal{M} \models \exists x\, (\varphi(x, \overline{a}) \wedge \forall y < x\, \neg\varphi(y, \overline{a}))$. But this witness must be unique, so the formula $\varphi(x, \overline{a}) \wedge \forall y < x\, \neg\varphi(y, \overline{a})$ must define an element of $\mathcal{M}$, say b. Hence, $b \in X$, so $\mathcal{A} \models \varphi(b, \overline{a}) \wedge \forall y < x\, \neg\varphi(b, \overline{a})$. It follows that $\mathcal{A}$ is an elementary substructure.

Since $\mathcal{A} \preccurlyeq \mathcal{M}$, $\mathcal{A} \models \mathsf{PA}$. But if $\mathcal{A}' \subset \mathcal{A}$ is a proper substructure, it cannot be elementary, as it will be missing elements which are required by the definable elements of $\mathcal{M}$. That is, there will be a formula $\varphi(x)$ such that $\mathcal{A} \models \exists x\, \varphi(x)$, but $\mathcal{A}' \models \neg\exists x\, \varphi(x)$. So $\mathcal{A}'$ couldn't satisfy exactly the same formulae as $\mathcal{A}$, so it couldn't be an elementary substructure. Thus, $\mathcal{A}$ is a nonstandard model of PA with no proper elementary substructure. □

4. Show that any theory with Skolem functions has quantifier elimination.

> ▶ **Answer:** Suppose a theory T has Skolem functions. So for every $\varphi(\overline{x}, y)$, there is a term $t_\varphi(\overline{x})$ such that $T \vdash \forall \overline{x}\ (\exists y\, \varphi(\overline{x}, y) \leftrightarrow \varphi(\overline{x}, t_\varphi(\overline{x}))$. But this is exactly what we need to prove quantifier elimination. Suppose we're trying to show that $\exists y\ \sigma(\overline{x}, y)$ is equivalent to a quantifier-free formula modulo T, where $\sigma(\overline{x}, y) := \bigwedge_i^n \varphi_i(\overline{x}, y) \wedge \bigwedge_j^k \neg\psi_j(\overline{x}, y)$, and where φ_i, ψ_j are all quantifier-free. Since T has Skolem functions, $T \vdash \exists y\ \sigma(\overline{x}, y) \leftrightarrow \sigma(\overline{x}, t_\sigma(\overline{x}))$ for some term t_σ. But then $\sigma(\overline{x}, t_\sigma(\overline{x}))$ is quantifier free, so $\sigma(\overline{x}, t_\sigma(\overline{x}))$ will work. □

5. We say a linear order $\langle L, < \rangle$ is **scattered** if for all $a < b < c$, the interval (a, b) either is empty or has a maximal element, and the interval (b, c) either is empty or has a minimal element. Let $\langle L, < \rangle$ be a scattered linear order. Show that $\mathrm{Th}\,(L, <)$ is recursive.

6. A linear order $\langle L, < \rangle$ is **anti-well-ordered** if every nonempty $X \subseteq L$ has a maximal element. Prove or disprove: if $\langle L, < \rangle$ is a countable anti-well-ordered linear order, then there is a countable anti-well-ordered linear order $\langle K, < \rangle$ such that $\langle L, < \rangle \equiv \langle K, < \rangle$.

> ▶ **Answer:** If $\langle L, < \rangle$ is an uncountable anti-well-ordered linear order, then by Downward Löwenheim-Skolem, there is a countable linear order $\langle K, < \rangle$ with $\langle K, < \rangle \preccurlyeq \langle L, < \rangle$ (and hence $\langle K, < \rangle \equiv \langle L, < \rangle$). Now, this doesn't immediately guarantee that K is anti-well-ordered, since there is no first-order sentence which is true exactly of the linear orders that are anti-well-ordered. But thankfully, this isn't an issue: if we take a subset $X \subseteq K$, then *a fortiori* $X \subseteq L$, and hence X has a maximal element. So consequently, K is anti-well-ordered as well. □

7. (a) Show that there is a pair of recursively inseparable r.e sets.

   (b) Show that any pair of disjoint Π^0_1 sets is recursively separable (i.e. not recursively inseparable).

▶ **Answer (a):** See this problem, page 5, part (a). The proof is basically the same, except you don't need to relativize to oracles, so C will be Δ^0_1, and A, B will be Σ^0_1. □

▶ **Answer (b):** First, we make the following claim:

Claim: If A, B are Σ^0_1 sets, then there are Σ^0_1 sets A', B' such that $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \varnothing$, and $A' \cup B' = A \cup B$.

▶ **Proof:** Enumerate A and B in stages, so that $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$. We construct A' and B' in stages alongside A and B. At stage 0, we let $A_0 = B_0 = A'_0 = B'_0 = \varnothing$. Now, suppose we've constructed A'_t and B'_t.

If $t = 2s$, then for stage $t+1$, we continue our construction of A to A_{s+1}. Let a_{s+1} be the new element added to A_{s+1}. If a_{s+1} is not in B_s, then set $A'_{t+1} = A'_t \cup \{a_{s+1}\}$. Otherwise, set $A'_{t+1} = A'_t$. In either case, take $B'_{t+1} = B'_t$.

If $t = 2s+1$, then for stage $t+1$, we continue our construction of B to B_{s+1}. Let b_{s+1} be the new element added to B_{s+1}. If b_{s+1} is not in A_{s+1}, then set $B'_{t+1} = B'_t \cup \{b_{s+1}\}$. Otherwise, set $B'_{t+1} = B'_t$. In either case, take $A'_{t+1} = A'_t$. It's easy to check that this construction works. □

Now, consider two disjoint Π^0_1 sets, A and B. Then their complements $\overline{A}$ and $\overline{B}$ are Σ^0_1. Hence, by the above claim, there are two sets $\overline{A'} \subseteq \overline{A}$ and $\overline{B'} \subseteq \overline{B}$ such that $\overline{A'} \cap \overline{B'} = \varnothing$ and $\overline{A'} \cup \overline{B'} = \overline{A} \cup \overline{B}$. Since $A \cap B = \varnothing$, $\overline{A} \cup \overline{B} = \mathbb{N}$, so $\overline{A'} \cup \overline{B'} = \mathbb{N}$. And since $\overline{A'}$ and $\overline{B'}$ were disjoint, $A' \cup B' = \mathbb{N}$. But $\overline{A'} = \mathbb{N} - \overline{B'}$, so:

$$A' = \overline{\mathbb{N} - \overline{B'}} = \overline{\mathbb{N} \cap B'} = \varnothing \cup \overline{B'} = \overline{B'}$$

(It's easier to see this if you draw out a Venn diagram; these calculations are also easy to figure out if you just remember that these set-theoretic operations correspond exactly to boolean operations.) Similarly, $B' = \overline{A'}$. Hence, both A' and B' are recursive. Furthermore, since $\overline{A'} \subseteq \overline{A}$, $A \subseteq A'$, and $A' \cap B = \varnothing$ since $A' \cap B' = \varnothing$ and $B \subseteq B'$. □

8. Let $\mathcal{L} = \langle Q, < \rangle$, and consider the structure $\mathbb{R}$ which interprets $<$ as the usual ordering and $Q^{\mathbb{R}} = \mathbb{Q}$. Find an axiomatization of $\mathsf{Th}(\mathbb{R}, <, Q)$, and show that it is complete.

▶ **Answer:** Let T be the theory:

$$\begin{aligned} T \;:=\; & \mathsf{DLO} \cup \{\forall x\, y\, (x < y \wedge Q(x) \wedge Q(y) \to \exists z\, (x < z \wedge z < y \wedge Q(z)))\} \\ & \cup \{\forall x\, \exists y\, (Q(y) \wedge x < y), \forall x\, \exists y\, (Q(y) \wedge y < x)\} \\ & \cup \{\forall x\, \exists y\, (\neg Q(y) \wedge x < y), \forall x\, \exists y\, (\neg Q(y) \wedge y < x)\} \end{aligned}$$

We'll show that T has quantifier elimination. Since the only atomic sentences of this language are $\top$ and $\bot$, it will follow that T is complete.

To show quantifier elimination, we consider the formula:

$$\exists x \left[\bigwedge_i y_i < x \wedge \bigwedge_j x < z_j \wedge (\neg) Q(x) \right]$$

We don't need to consider the negated atomics, $y_n \nless x$ and $x \nless z_m$, since these are equivalent to $y_n \geqslant x$ and $x \geqslant z_m$, which we can eliminate. Furthermore, according to T, if $\exists x \left[\bigwedge_i y_i < x \wedge \bigwedge_j x < z_j \right]$, then there's a witness that's rational (since this formula only references finitely many y_is and z_js, and since Q according to T is both dense and coinitial/cofinal with $\mathbb{R}$). Similarly, there will be such a number that is irrational. Hence, we can just consider the formula:

$$\exists x \left[\bigwedge_i y_i < x \wedge \bigwedge_j x < z_j \right]$$

But then by quantifier elimination of DLO, this is equivalent to just $\bigwedge_{i,j} y_i < z_j$. Hence, T has quantifier elimination. □

9. Prove that there are sets A, B such that $A \nleqslant_T B$ and $B \nleqslant_T A$.

▶ **Answer:** See this problem, page 8. Since you don't need the sets to be r.e, the shorter Kleene-Post proof suffices. □

June 2008

1. Prove or refute:

   (a) If A and B are Σ^0_1, then there's a Δ^0_1 set C that separates them.

   (b) If A and B are Π^0_2, then there's a Δ^0_2 set C that separates them.

▶ **ANSWER (a,b):** See this problem, page 83. For part (b) of this problem, use part (b) of that problem, except relativize everything to an oracle. □

2. Let T be a decidable theory in a finite language with no finite models. Show that T has a model $\mathcal{A}$ with universe $\mathbb{N}$ such that $\{\langle\varphi(\overline{x}), \overline{a}\rangle \mid \mathcal{A} \models \varphi(\overline{a})\}$ is recursive.

▶ **ANSWER:** We proceed by simply constructing the canonical model for T, and show that at every step of the construction is decidable. We'll achieve a complete theory Σ which yields a countable canonical model (so the unvierse could just be $\mathbb{N}$). And since the model is canonical, every object is named by a closed term t. Hence, satisfaction of formulae reduces to satisfaction of sentences.

First, we add constants $c_0, c_1, c_2, \ldots$ to the language and enumerate the sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$ of the expanded language. At stage 0, we set $\Sigma_0 := T$. At stage $n+1$, check to see if $\Sigma_n \cup \{\varphi_n\}$ is consistent. This can be done recursively since $\Gamma_n := \Sigma_n - T$ is finite and T is decidable (so you can just search for a proof of $\bigwedge \Gamma_n \wedge \varphi_n \to \perp$ from T). If it isn't consistent, set $\Sigma_{n+1} := \Sigma_n$. Otherwise, if $\varphi_n := \exists x\, \psi(x)$ for some ψ, then pick the least c_i not yet used in Σ_n, and set $\Sigma_{n+1} := \Sigma_n \cup \{\varphi_n, \psi(c_i)\}$. Otherwise, set $\Sigma_{n+1} := \Sigma_n \cup \{\varphi_n\}$. Finally, set $\Sigma_\omega := \Sigma := \bigcup_i \Sigma_i$.

Since each step in this procedure is decidable, we'll get a canonical model $\mathcal{A}$ where $\mathsf{Th}(\mathcal{A}) = \Sigma_\omega$, which is decidable. And since $\mathcal{A}$ is the canonical model, every element of $\mathcal{A}$ is denoted by a closed term of the expanded language. Hence, satisfaction in $\mathcal{A}$ reduces to the truth of sentences (in the expanded language) in $\mathcal{A}$, so $\{\langle\varphi(\overline{x}), \overline{a}\rangle \mid \mathcal{A} \models \varphi(\overline{a})\}$ is decidable. □

3. Show that there is a nonstandard $\mathcal{M} \models \mathsf{PA}$ and a nonstandard $a \in M$ such that a is definable in $\mathcal{M}$.

▶ **Answer:** Let $\mathcal{M} \models \mathsf{PA} + \neg\mathsf{Con}\,(\mathsf{PA})$. Then there's a nonstandard $a \in M$ such that $\mathcal{M} \models \mathsf{Prf}_{\mathsf{PA}}\,(a, \ulcorner\bot\urcorner)$. Since for any $\varphi(x)$, $\mathsf{PA} \vdash \exists x\, \varphi(x) \to \exists x\, (\varphi(x) \wedge \forall y < x\, \neg\varphi(y))$, it follows there must be a least such $a \in M$. But then that least such a will be defined by $\varphi(x) \wedge \forall y < x\, \neg\varphi(x)$. □

4. Let $\mathcal{A}$ be a model, and let $a, b \in A$ be two distinct elements. Show that the following are equivalent:

   (a) There is a definable function f for which $f(a) = b$.

   (b) For any $\mathcal{B} \succcurlyeq \mathcal{A}$, and any automorphism $\sigma : \mathcal{B} \to \mathcal{B}$, if $\sigma(a) = a$, then $\sigma(b) = b$.

▶ **Answer:**

(a) ⇒ (b): Suppose there is such an f that is definable, say by formula $\varphi(x, y)$. Suppose also that $\mathcal{B} \succcurlyeq \mathcal{A}$ and that $\sigma : \mathcal{B} \to \mathcal{B}$ is an automorphism that fixes a. Then:

$$\begin{aligned} \mathcal{B} \models f(a) = b &\leftrightarrow \varphi(a, b) \\ &\leftrightarrow \varphi(\sigma(a), \sigma(b)) \\ &\leftrightarrow \varphi(a, \sigma(b)) \\ &\leftrightarrow f(a) = \sigma(b) \end{aligned}$$

Thus, $\sigma(b) = b$. □

(b) ⇒ (a): Suppose there is no definable function f sending $a \mapsto b$. Add a new constant c to the language, and define the theory:

$$T := \mathsf{ElDiag}\,(\mathcal{A}) \cup \{b \neq c\} \cup \{\varphi(a, c) \mid \mathcal{A} \models \varphi(a, b)\}$$

Claim: T is finitely satisfiable.

▶ **PROOF:** Suppose not. Then there is a list of some $\varphi_1(a,c),\ldots,\varphi_n(a,c)$ where $\mathsf{ElDiag}(\mathcal{A}) \vdash \bigwedge_i \varphi_i(a,c) \to b = c$. Hence, $\mathsf{ElDiag}(\mathcal{A}) \vdash \forall z\ (\bigwedge_i \varphi_i(a,z) \to b = z)$. Now, let:

$$\psi(x,y) := \bigwedge_i \varphi_i(x,y) \wedge \forall z \left(\bigwedge_i \varphi_i(x,z) \to y = z\right)$$

By hypothesis, $\mathcal{A} \models \bigwedge_i \varphi_i(a,b)$, so $\mathsf{ElDiag}(\mathcal{A}) \vdash b = c \to \bigwedge_i \varphi_i(a,c)$. Hence, $\mathsf{ElDiag}(\mathcal{A}) \vdash b = c \to \psi(a,c)$, since $\mathsf{ElDiag}(\mathcal{A}) \vdash \forall z\ (\bigwedge_i \varphi_i(a,z) \to b = z)$. Furthermore, $\mathsf{ElDiag}(\mathcal{A}) \vdash \psi(a,c) \to b = c$, since trivially $\mathsf{ElDiag}(\mathcal{A}) \vdash \psi(a,c) \to \bigwedge_i \varphi_i(a,c)$. Finally, $\mathsf{ElDiag}(\mathcal{A}) \vdash \forall x,y,y'\ (\psi(x,y) \wedge \psi(x,y') \to y = y')$, since $\psi(x,u)$ implies $\bigwedge_i \varphi_i(x,u)$, so if $\psi(x,y)$ and $\psi(x,y')$, $\bigwedge_i \varphi_i(x,y) \wedge \bigwedge_i \varphi_i(x,y')$, which implies $y = y'$. Hence, $\psi(x,y)$ defines a function f where $f(a) = b$, $\perp$. □

Hence, T is satisfiable by compactness. So there's an elementary extension $\mathcal{B} \succcurlyeq \mathcal{A}$ such and an automorphism $\sigma : \mathcal{B} \to \mathcal{B}$ such that $\sigma(a) = a$ and $\sigma(b) = c$, with $c \neq b$. □

5. Let $\mathcal{L} = \{U, V\}$, where U, V are unary predicates. Describe all the complete theories of $\mathcal{L}$. Show that they are distinct and exhaust all the possibilities.

▶ **ANSWER:** The four things that such a complete $\mathcal{L}$-theory needs to specify are the number of elements in $U \wedge V$, $U \wedge \neg V$, $\neg U \wedge V$, and $\neg U \wedge \neg V$. The theory in any particular case can either say that there are exactly n-many things for some particular $n \in \omega$, or that there are at least n-many things for all $n \in \omega$.

To show that any such theory T is complete, note that there are only two cases: either T has only infinite models, or T has only finite models of a certain size k. If the former, then any two models of T will be isomorphic, since we can build the isomorphism just be sending each element in $U \wedge V$ of the first model to a unique element in $U \wedge V$ in the second, thus producing a bijection.

If T only has infinite models, then by Downward Löwenheim-Skolem it has countable models. But any two countable models will also be isomorphic: just build the isomorphism as before, knowing that if any case above is infinite, then the fact that the model is countable will guarantee that you can still build a bijection between the two models. Hence, T is ω-categorical, and so complete by Vaught's test. $\square$

6. Show that Fin is Σ^0_2-complete.

▶ **ANSWER:** To show that Fin is Σ^0_2:

$$\begin{aligned}
\mathsf{Fin}(e) &\Leftrightarrow W_e \text{ is finite} \\
&\Leftrightarrow \exists s\,, n\, [\mathsf{Seq}\,(s) \wedge \mathsf{lh}\,(s) = n \wedge \forall i, j < n\, ((s)_i \neq (s)_j) \wedge \\
&\qquad\qquad \forall x\, (x \in W_e \rightarrow \exists i < n\, ((s)_i = x))] \\
&= \exists s, n\, (\Delta^0_1 \wedge \Delta^0_1 \wedge \Delta^0_1 \wedge \forall x\, (\Sigma^0_1 \rightarrow \Delta^0_1)) \\
&= \Sigma^0_2
\end{aligned}$$

To show it's Σ^0_2-hard, let $A(e)$ iff $\exists x\, \forall y\, R(e, x, y)$, where R is Δ^0_1. Define:

$$h(e, x) = \begin{cases} 1 & \text{if } \forall u \leqslant x\, \exists y\, \neg R(e, u, y) \\ \uparrow & \text{otherwise} \end{cases}$$

This is recursive, since R is Δ^0_1, so let $h(e, x) = \phi_{s(x)}(y)$. Then:

$$\begin{aligned}
e \in A &\Rightarrow \exists x\, \forall y\, R(e, x, y) &&\Rightarrow \exists x\, \forall z > x\, (h(e, z) \uparrow) &&\Rightarrow s(e) \in \mathsf{Fin} \\
e \notin A &\Rightarrow \forall x\, \exists y\, \neg R(e, x, y) &&\Rightarrow \forall x\, (h(e, x) \downarrow) &&\Rightarrow s(e) \notin \mathsf{Fin}
\end{aligned}$$

This completes the reduction. $\square$

7. Let $\mathcal{A} = \langle A, I, f, g, \ldots \rangle$ be a structure in a finite language $\mathcal{L}$, where I is a unary predicate and f, g are binary functions. Let π be an isomorphism between $\langle \mathbb{N}, +, \times \rangle$ and $\langle I, f \restriction I, g \restriction I \rangle$. Show that $\{\pi(\ulcorner \varphi \urcorner) \mid \mathcal{A} \models \varphi\}$ is not definable over $\mathcal{A}$ without parameters.

▶ **Answer:** I will simplify the notation and let $\mathcal{A} := \langle A, N, +, \times, \ldots \rangle$ with $\mathbb{N} \subseteq A$. Since $\mathcal{L}$ is finite, the natural numbers can still code up formulae from $\mathcal{L}$, so we'll let $\ulcorner \varphi \urcorner$ denote the gödel number in this new coding system of φ. The result, then, is that we want to show that $\{\ulcorner \varphi \urcorner \mid \mathcal{A} \models \varphi\}$ is not definable (without parameters).

Suppose it were definable, let's say by the formula $\tau(x)$. In other words, $\mathcal{A} \models \tau(\ulcorner \varphi \urcorner) \Leftrightarrow \mathcal{A} \models \varphi$ (do you smell a Liar?). Suppose that something like the Fixed-Point Lemma holds for $\mathcal{A}$—that is, suppose for every formula $\varphi(x)$, there's a sentence δ such that $\mathcal{A} \models \delta \leftrightarrow \varphi(\ulcorner \delta \urcorner)$. Then it follows that there's a sentence λ such that $\mathcal{A} \models \lambda \leftrightarrow \neg\tau(\ulcorner \lambda \urcorner)$. But since $\mathcal{A} \models \lambda \leftrightarrow \tau(\ulcorner \lambda \urcorner)$, $\mathcal{A} \models \tau(\ulcorner \lambda \urcorner) \leftrightarrow \neg\tau(\ulcorner \lambda \urcorner)$, $\bot$.

Claim: For any formula $\varphi(x)$, there is a sentence δ such that $\mathcal{A} \models \delta \leftrightarrow \varphi(\ulcorner \delta \urcorner)$.

▶ **Proof:** Since coding for $\mathcal{L}$ can still be done recursively on $\mathbb{N}$, define the following recursive function on $\mathbb{N}$:

$$\mathsf{sub}(n, v) = \begin{cases} \ulcorner \theta(\underline{n}) \urcorner & \text{if } v = \ulcorner \theta(x) \urcorner \\ 0 & \text{otherwise} \end{cases}$$

Now, $\mathbb{N} \models \mathsf{sub}(n, \ulcorner \theta \urcorner) = z \leftrightarrow z = \ulcorner \theta(\underline{n}) \urcorner$ for all θ and n. Hence, $\mathcal{A} \models N(n) \rightarrow (\mathsf{sub}(n, \ulcorner \theta \urcorner) = z \leftrightarrow z = \ulcorner \theta(\underline{n}) \urcorner)$, i.e. $\mathcal{A} \models \mathsf{sub}(n, \ulcorner \theta \urcorner) = z \leftrightarrow z = \ulcorner \theta(\underline{n}) \urcorner$ for all $n \in \mathbb{N}$. Now define:

$$\begin{aligned} \alpha(x) &:= \exists y \ (\mathsf{sub}(x, x) = y \wedge \varphi(y)) \\ \delta &:= \alpha(\ulcorner \alpha \urcorner) \end{aligned}$$

Then:

$$\begin{aligned} \mathcal{A} \models \delta &\leftrightarrow \exists y \ (\mathsf{sub}(\ulcorner \alpha \urcorner, \ulcorner \alpha \urcorner) = y \wedge \varphi(y)) \\ &\leftrightarrow \exists y \ (y = \ulcorner \alpha\,(\ulcorner \alpha \urcorner) \urcorner \wedge \varphi(y)) \\ &\leftrightarrow \varphi(\ulcorner \delta \urcorner) \end{aligned}$$

Hence $\mathcal{A} \models \delta \leftrightarrow \varphi(\ulcorner \delta \urcorner)$. □

This completes the proof. Note that it works because the proof of the Fixed Point Lemma doesn't depend on φ being in the language of PA: we don't use anything about φ for the proof. □

8. Give an example of first-order languages $\mathcal{L} \subseteq \mathcal{L}'$ and complete theories $T \subseteq T'$ with T in $\mathcal{L}$ and T' in $\mathcal{L}'$ such that T' is $\aleph_1$-categorical, but T is not. Show that your example works.

▶ **ANSWER:** Let $\mathcal{L} = \{E\}$, where E is a binary predicate, and let T be the theory stating that E is an equivalence relation, and that there are exactly two equivalence classes, both of which are infinite. T is $\aleph_0$-categorical, since if $\mathcal{A}$ is a countable model of T, its two equivalence classes must both be countably infinite, so there will be a bijection between those equivalence calsses and those of any other countable model. Hence, by Vaught's test, T is complete. However, T is not $\aleph_1$-categorical. One uncountable model of T is the one with two uncountable equivalence classes; another nonisomorphic model is one with an uncountable class and a countable class.

Now, let $\mathcal{L}' = \{E, f, c\}$, where f is unary function symbol and c is a constant, and let T' be like the theory T except it states that f is a bijection between the two classes, with c being in the domain of f. That is, T' will contain:

- T
- "f is a bijection"
- "For all x, if $E(x, c)$, then $f(x) = y$ where $\neg E(x, y)$. Otherwise, $f(x) = x$"

We can extend the reasoning as before in T to show that T' is $\aleph_0$-categorical. One just simply has to make sure that, when building the isomorphism h, $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$, and that $h(f^{\mathcal{A}}(a))$ maps to $f^{\mathcal{B}}(h(a))$, which is easy enough. Hence, T' is complete. But now, in fact, it is also $\aleph_1$-categorical, since there must be a bijection between the two equivalence classes, so the model of T where one class is countable is no longer a model of T'. □

January 2008

1. Prove or disprove: if $\mathcal{M} \models \mathsf{PA}$ is nonstandard, and $a \in M$ is nonstandard, then a is not definable.

> ▶ **Answer:** False. See here, page 87. □

2. Show that Rec is Σ^0_3-complete.

> ▶ **Answer:** The proof is rather involved, and I won't give it here. See Soare [6] for a proof. It showing that $(\Sigma^0_3, \Pi^0_3) \leqslant_1 (\mathsf{Cof}, \mathsf{Cpl})$, where Cpl is the set of indices that are Turing equivalent to K. There's also a proof in Rogers [5] that uses a priority argument directly. □

3. Let $\mathcal{L}$ be a countable language, and let T be a theory with infinite models. Show that there is a model of size ω_1 in which at most ω-many 1-types are realized.

> ▶ **Answer:** Let T^* be the skolemization of T, and let $\mathcal{A} \models T^*$ be countable.[a] Clearly, $\mathcal{A}$ is a model which can only realize countably many 1-types.
>
> Let $\mathcal{M}$ be an Ehrenfeucht-Mostowski model of $\mathsf{ElDiag}(\mathcal{A})$ with spine $\langle \omega_1, < \rangle$. Consider $\mathcal{B} := \mathrm{Hull}(\{c_\alpha \mid \alpha \in \omega_1\})$. $\mathcal{B}$ still satisfies the same quantifier-free formulae over A as $\mathcal{A}$ (since the elements of a were denoted by constants in $\mathsf{ElDiag}(\mathcal{A})$, and thus included in the generated substructure), so $\mathcal{B} \succcurlyeq \mathcal{A}$ since T^* is a Skolem theory. Hence, the elements of A all still satisfy the same 1-types, so it suffices to check that the elements of $B - A$ only satisfy countably many more new 1-types.
>
> Notice that the c_αs all satisfy the same 1-type (since each c_α is an order-indiscernible sequence of length one). Hence, at most one more 1-type could have been realized by them. So it suffices to check that only countably many more 1-types are realized by elements of the form $t(\overline{c}, \overline{a})$, where t is a term and $\overline{a} \in A$. And since there are only countably terms (countable language) and only countably many finite sequences of elements from A (A is countable), it suffices to check that fixing t

and $\overline{a}$, there are only countably 1-types realized by elements of the form $t(\overline{c}, \overline{a})$ (since then there would only be at most $\aleph_0^3 = \aleph_0$ many more 1-types realized).

Since $\overline{c}_\alpha$s are order-indiscernibles, only the relative order of $\overline{c}$ matters. That is, if $t(\overline{x}, \overline{a})$ is our term, and the length of $\overline{x}$ is n, there are only at most $n!$ different new 1-types that could be realized as a result of plugging in the order-indiscernible constants for $\overline{x}$.[b,c] Since this applies for each $n \in \omega$, that means that there can only be $\aleph_0$ many more 1-types realized by elements of the form $t(\overline{c}, \overline{a})$.

Hence, there are only at most $\aleph_0$ many 1-types realized in $\mathcal{B}$. □

[a] The skolemized language is still countable, so this is allowed by Downward Löwenheim-Skolem.

[b] It might help to give an example. Suppose the term $t(x_1, x_2, x_3, \overline{a})$ has 3 opens slots. Then there are at most six more new 1-types that could be realized, based on the following terms:

- $t(c_1, c_2, c_3, \overline{a})$
- $t(c_1, c_3, c_2, \overline{a})$
- $t(c_2, c_1, c_3, \overline{a})$
- $t(c_2, c_3, c_1, \overline{a})$
- $t(c_3, c_1, c_2, \overline{a})$
- $t(c_3, c_2, c_1, \overline{a})$

Since any other sequence of length three is order-indiscernible, it doesn't matter which constants appear instead of c_1, c_2, c_3. For instance, the 1-type realized by $t(c_1, c_3, c_2, \overline{a})$ is the same as the type realized by $t(c_5, c_\omega, c_{20}, \overline{a})$ since $c_5 < c_{20} < c_\omega$ is the same relative order as $c_1 < c_2 < c_3$.

[c] The way EM models are constructed in Hodges [2], this isn't strictly necessary apparently: there should only be one n-type realized by each of these terms. But I'm following Marker [4] here.

4. (a) Show that there is a pair of recursively inseparable r.e sets.

   (b) Show that any pair of disjoint Π^0_1 sets is recursively separable.

▶ **Answer (a,b):** See this problem, page 5, for part (a), and this problem, page 83, for part (b). □

5. Let $\mathcal{L} = \{U\}$, where U is a unary predicate. Prove or disprove: the set of validities in $\mathcal{L}$ is recursive.

▶ **Answer:** True. We'll prove this by showing that this language has the **finite model property**: that is, if an $\mathcal{L}$-sentence φ is satisfiable, then it has a finite model. In fact, we'll be able to show that it has a model of size no greater than $2n$, where n is the number of variables occurring in φ. It will follow that we can determine whether φ is valid just be running through the finitely many finite models of size no greater than $2n$ and just check whether $\neg\varphi$ holds in any of them.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure, with some tuple of elements $\overline{a}, \overline{b} \in A$ of the same length. We'll say that $\overline{a}$ and $\overline{b}$ **match** just in case:

(i) For each i, $\mathcal{A} \models U(a_i)$ iff $\mathcal{A} \models U(b_i)$

(ii) For each i, j, $a_i = a_j$ iff $b_i = b_j$.

Now, let n be the number of variables occurring in φ. If $\mathcal{A}$ has no more than n-many elements satisfying $(\neg)U(x)$, then let $B_+(B_-)$ be the subset of A containing those elements; otherwise, let $B_+(B_-)$ contain exactly n-many elements satisfying $(\neg)U(x)$. Let $B = B_+ \cup B_-$, and define a model $\mathcal{B} := \langle B, U \upharpoonright B\rangle$. Notice that $\mathcal{B}$ is of size no greater than $2n$.

Claim: $\mathcal{B} \models \varphi$.

▶ **Proof:** We will proceed by induction on the complexity of subformulae in φ. We'll show that if $\overline{a} \in A$ and $\overline{b} \in B$ match, then for any subformula $\psi(\overline{x})$ of φ (with say $k \leqslant n$ free variables, possibly $k = 0$), $\mathcal{A} \models \psi(\overline{a})$ iff $\mathcal{B} \models \psi(\overline{b})$.

Atomic: Either $\psi(x)$ is $U(x)$ or $\psi(x, y)$ is $x = y$. In the former case, since a matches b, and since $U^B = U^A \restriction B$, $\mathcal{A} \models U(a)$ iff $\mathcal{B} \models U(b)$. In the latter case, since a_1, a_2 matches b_1, b_2, then $a_1 = a_2$ iff $b_1 = b_2$. ✓

Boolean Combos: Straightforward. ✓

Existential: $\psi(\overline{x})$ is $\exists y\ \theta(\overline{x}, y)$, where the inductive hypothesis holds for $\theta(\overline{x}, y)$. Then $\mathcal{A} \models \psi(\overline{a})$ iff for some $c \in A$, $\mathcal{A} \models \theta(\overline{a}, c)$. Note that $k < n$, where k is the length of $\overline{x}$. By inductive hypothesis, if there is a d such that $\overline{a}, c$ matches $\overline{b}, d$, then we'll have $\mathcal{A} \models \theta(\overline{a}, c)$ iff $\mathcal{B} \models \theta(\overline{b}, d)$. So it suffices to show that there is a d such that $\overline{a}, c$ matches $\overline{b}, d$.

If $c = a_i$ for some $i \leqslant k$, then we can just set $d = b_i$ as well. Otherwise, suppose WLOG that $\mathcal{A} \models U(c)$ (the same reasoning applies to $\neg U(x)$). Either $|U^{\mathcal{A}}| \leqslant n$, or $|U^{\mathcal{A}}| > n$. If $|U^{\mathcal{A}}| = m \leqslant n$, then by construction $U^{\mathcal{B}} = U^{\mathcal{A}}$. Since there are strictly less than m-many of these elements among $\overline{a}$, there will be strictly less than m-many of these elements among the matching $\overline{b}$, and hence there will be another $d \in U^{\mathcal{B}}$ not among $\overline{b}$. If $|U^{\mathcal{A}}| > n$, then there will be exactly n-many things in $U^{\mathcal{B}}$. But since $k < n$, there will still be a $d \in U^{\mathcal{B}}$ not among $\overline{b}$ (since there's only k-many). Either way, there's a d such that $\overline{a}, c$ matches $\overline{b}, d$. ✓

Hence, by induction, $\mathcal{A} \models \varphi(\overline{a})$ iff $\mathcal{B} \models \varphi(\overline{b})$. □

Since $\mathcal{B}$ was of size at most $2n$, this completes the proof.[10] □

[10] The proof could easily be extended to show that monadic first-order logic is decidable. If $\mathcal{L}$ had, say, finitely many unary predicates $P_1(x), \ldots, P_m(x)$, then by revising the first clause in the definition of matching to say $\mathcal{A} \models P_j(a_i)$ iff $\mathcal{A} \models P_j(b_i)$ for all $j \leqslant m$, the proof should go through as expected, but now the size of your model needs to be $n \cdot 2^k$, where k is the number of predicates

6. Show that the class of existentially closed groups is not first-order axiomatizable.

7. Suppose f is a total recursive function. Prove or disprove:

   (a) There is an e such that $W_{f(e)} = \{e\}$.

   (b) There is an e such that $W_e = \{f(e)\}$.

▶ **Answer (a):** False. We'll construct a counterexample. Let:

$$h(e,x) = \begin{cases} 1 & \text{if } x \leqslant e+1 \\ \uparrow & \text{otherwise} \end{cases}$$

Clearly, h is recursive, so there's a total recursive function f such that $h(e,x) = \phi_{f(e)}(x)$. But then $W_{f(e)} = \{0, \ldots, e+1\}$ for all e. So $|W_{f(e)}| \geqslant 2$ for all e, and hence there can be no such e where $W_{f(e)} = \{e\}$. □

▶ **Answer (b):** True. Define:

$$h(e,x) = \begin{cases} 1 & \text{if } x = f(e) \\ \uparrow & \text{otherwise} \end{cases}$$

Clearly h is recursive, and so there is a total recursive s such that $h(e,x) = \phi_{s(e)}(x)$. That is, $W_{s(e)} = \{f(e)\}$ for all e. But then, by the Recursion theorem, there's a d such that $W_{s(d)} = W_d = \{f(d)\}$. □

8. Let T be a $\aleph_0$-categorical theory in a countable language $\mathcal{L}$, and let $\mathcal{A} \models T$ be countably infinite. Determine the cardinality of the automorphism group of $\mathcal{A}$. Prove you're right.

---

occurring in φ. You'll then need to ensure that every combination of literals among the P_is have at most n-many elements in $\mathcal{B}$. This could easily extend further for the case where you have countably infinitely many unary predicates.

▶ **ANSWER:** We argue that the cardinality of the automorphism group must be $2^{\aleph_0}$ as follows. Let T^* be the skolemization of T and let $\mathcal{A}^*$ be the model expanded from $\mathcal{A}$. Note that $\mathcal{A}^*$ is still countable.

Consider a countable Ehrenfeucht-Mostowski model $\mathcal{M}$ of $\mathsf{ElDiag}(\mathcal{A}^*)$ whose spine is $\omega \times \mathbb{Q}$ (with lexicographical ordering), and consider $\mathcal{B} := \mathrm{Hull}(\{c_a \mid a \in \omega \times \mathbb{Q}\})$. $\mathcal{B}$ is still countable, and still satisfies T^*, since in Skolem theories every substructure is elementary. Hence, $\mathcal{B} \upharpoonright \mathcal{L} \cong \mathcal{A}$, so it suffices to show that there are $2^{\aleph_0}$ automorphisms on $\mathcal{B}$ (as those will correspond to automorphisms on $\mathcal{A}$).

For every $X \subseteq \omega$, we define a mapping σ_X on $\omega \times \mathbb{Q}$ as follows:

$$\sigma_X(\langle n, q\rangle) = \begin{cases} \langle n, q+1\rangle & \text{if } n \in X \\ \langle n, q\rangle & \text{otherwise} \end{cases}$$

That is, σ_X just shifts the elements in the $\mathbb{Q}$-chains associated with elements in X. This preserves the order of $\omega \times \mathbb{Q}$, and so is an automorphism on σ_X. Since automorphisms on the order extend to automorphisms on the model, and since there are $2^{\aleph_0}$-many distinct subsets of ω, it follows that there are $2^{\aleph_0}$-many distinct automorphisms on $\mathcal{B}$. □

9. Prove or disprove: addition is definable in $\langle \mathbb{Q}, \cdot \rangle$.

▶ **ANSWER:** False. We will show that there is an automorphism σ (i.e. a map preserving multiplication) that does not preserve addition as follows. If $q \in \mathbb{Q}$ is reduced form can be written as $q = m/n$, where $m, n \in \mathbb{N}$ and the prime factorization of m is $m = 2^{k_1} \cdot 3^{k_2} \cdot \dots \cdot p_i^{k_i}$, then our mapping will sent $q \mapsto m'/n$ where $m' = 2^{k_2} \cdot 3^{k_1} \cdot \dots \cdot p_i^{k_i}$ (that is, just switch the exponents of 2 and 3). Then:[a]

$$
\begin{aligned}
\sigma(q \cdot r) &= \sigma\left(\frac{m_q}{n_q} \cdot \frac{m_r}{n_r}\right) \\
&= \sigma\left(\frac{2^{k_1} \cdot 3^{k_2} \cdot \cdots \cdot p_i^{k_i}}{n_q} \cdot \frac{2^{l_1} \cdot 3^{l_2} \cdot \cdots \cdot p_j^{l_j}}{n_r}\right) \\
&= \sigma\left(\frac{2^{k_1+l_1} \cdot 3^{k_2+l_2} \cdot \cdots \cdot p_i^{k_i+l_i} \cdot \cdots \cdot p_j^{l_j}}{n_q \cdot n_r}\right) \\
&= \frac{2^{k_2+l_2} \cdot 3^{k_1+l_1} \cdot \cdots \cdot p_i^{k_i+l_i} \cdot \cdots \cdot p_j^{l_j}}{n_q \cdot n_r} \\
&= \frac{2^{k_2} \cdot 3^{k_1} \cdot \cdots \cdot p_i^{k_i}}{n_q} \cdot \frac{2^{l_2} \cdot 3^{l_1} \cdot \cdots \cdot p_j^{l_j}}{n_r} \\
&= \sigma\left(\frac{2^{k_1} \cdot 3^{k_2} \cdot \cdots \cdot p_i^{k_i}}{n_q}\right) \cdot \sigma\left(\frac{2^{l_1} \cdot 3^{l_2} \cdot \cdots \cdot p_j^{l_j}}{n_r}\right) \\
&= \sigma\left(\frac{m_q}{n_q}\right) \cdot \sigma\left(\frac{m_r}{n_r}\right) \\
&= \sigma(q) \cdot \sigma(r)
\end{aligned}
$$

Hence σ preserves multiplication, and thus is an automorphism on $\langle \mathbb{Q}, \cdot \rangle$. But it doesn't preserve addition: for instance, $\sigma(2+2) = \sigma(4) = \sigma(2^2) = 3^2 = 9$, whereas $\sigma(2) + \sigma(2) = 3 + 3 = 6$. □

[a] WLOG, assume $i < j$ here.

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